Earthquake risk: Including Uncertainties in the Ground Motion Calculations

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Executive Summary

Earthquake risk models used by the insurance industry for estimating the damage caused by an event typically use the mean of the ground motion intensity to predict the damage to a particular building. On the other hand, the ground motion intensity can be modelled as a random variable. Literature related to ground-motion predictive equation derivation includes, as a rule, the estimate of the standard deviation associated with the distribution of the intensity. The Study Group was asked to find a way to include the uncertainty associated with the prediction of the ground motion intensity contained in the standard deviation into the damage calculation, in a way in which the computational effort is not increased significantly

The Study Group proposed a way forward based on Bayes' theorem for the marginal distribution of damage and found an analytical expression for the damage distribution function. However, the expression is an integral that needs to be evaluated numerically and the Gaussian-Hermite quadrature was proposed to carry out the calculations. The approach seems plausible to be included in the existing models and the additional computational load is estimated as to be marginal relating to the current computational demands.

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1 Introduction

1.1 Catastrophe modelling

(1.1.1) AIR Worldwide is a catastrophe risk modelling company who provides their clients in insurance and re-insurance markets with models that allow earthquake damages, within other perils, to be estimated. Part of the process carried out on earthquake risk assessment can be summarised, at a high level, as presented below.

- A given geographical location, that could be a square with a side of 250 km is represented as a number of points on the grid. A typical discretisation may use a resolution of 50 meters distance between points in both latitude and longitude directions.
- For each earthquake event with magnitude M in the stochastic catalogue, the induced ground motion intensities are calculated at each point on the grid. A typical stochastic catalogue contains hundreds of thousands of events.
- At each point and for each earthquake, Ground Motion Prediction Equations (GMPEs) are used to calculate the motion intensity (*e.g.* Peak Ground Acceleration) y. A typical GMPE has a form

$$\log_{10} y = c_1 + c_2 M + c_3 \log_{10}(R + c_4) + c_5 S + \sigma_{GMPE}, \quad (1)$$

  where y is the intensity of ground motion, M is the magnitude, R is the source-to-site distance, S is the site factor, which depends on the type of soil, and σ_{GMPE} is the standard deviation. In order to capture the epistemic uncertainty in ground motion prediction, a number of equations are used in a logic tree framework. For a given case, seven GMPEs are used to calculate the intensity at each point. [1] A weighted averaged value of y is used for subsequent calculations.
- Finally, the value of y is used to estimate the damage. In Figure 1 the blue curve is the mean of the β-distribution (red curve) used to model the damage distribution for a given y. Each β curve corresponds to a single value of the ground motion intensity y.

It is worth pointing out that the calculations are very computationally demanding. For instance, the earthquake catalogue for a given region contains about 900,000 events, and the ground motions induced by each event are calculated for the entire geographical grid previously mentioned (i.e., 25,000,000 points in the grid). Then 7 equations are used at each point, which overall gives over 150 trillion calculations before averaging or calculating the damage resulting from all simulated earthquakes for all points on the grid.

[1] Please see exact examples of GMPEs and their derivation in references [1] and [2].

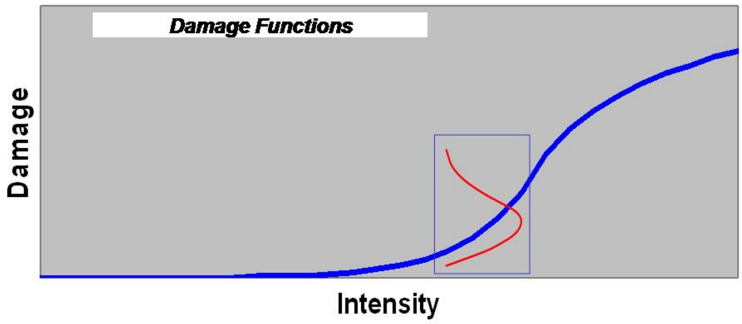

Figure 1: Mean value of damage is plotted (blue curve) for each value of mean ground motion (intensity) y. Damage is modelled to have a β-distribution (red curve) around its mean.

1.2 Problem

(1.2.1) Equation 1 includes σ_{GMPE}, a standard deviation that describes uncertainties associated with a particular type of GMPE. The most recent equations present σ_{GMPE} as a function of earthquake magnitude M and site properties. As an approximation to the distribution of ground motion intensity, the mean value of intensity from each GMPE is used.

(1.2.2) AIR would like to investigate whether the information about a distribution of ground motion intensity described by σ_{GMPE} can be incorporated in the damage calculations without significant increment in the computational effort.

2 The work of the Study Group

2.1 Mathematical formulation of the problem

(2.1.1) First, let us formulate the problem in mathematical terms. Let Y be the random variable describing the Ground Motion Intensity. Let Y to be distributed as log-normal[2]: $Y \sim N(\log y; m, \sigma)$, where m is the mean value of $logy$ distribution, σ is its standard deviation[3] and y is a particular realisation of Y.

(2.1.2) For each value y of Y the damage caused by this intensity is a conditional probability $D|\{Y = y\}$. It is β-distributed with mean μ, which depends on y, $\mu = \mu(y)$, and standard deviation, which also depends on μ.

[2]Please note that here and everywhere else in the text where the base of logarithm is not specified, we mean the natural logarithm (note that logarithm base 10 is used in GMPE).

[3]Please note that σ is different from σ_{GMPE} as the former is defined for the log-normal distribution while the latter is specified for the $\log_{10}$ distribution. The relationship is $\sigma = \log(10)\sigma_{GMPE}$.

(2.1.3) The problem is to find the marginal distribution of D. To clarify, let us look again at the vulnerability curve at the damage *vs* intensity graph.

(2.1.4) Figure 2 shows $\mu(y)$, which is the vulnerability curve, *i.e.* it gives the mean value of damage for any given intensity. A vulnerability curve is a continuous, monotonically increasing function bounded between 0 and 1 that gives the damage to a particular structure given the ground motion at that structure. There are different vulnerability curves for different types of buildings and for each building type and each intensity y, $\mu(y)$ is the mean of the damage extent.

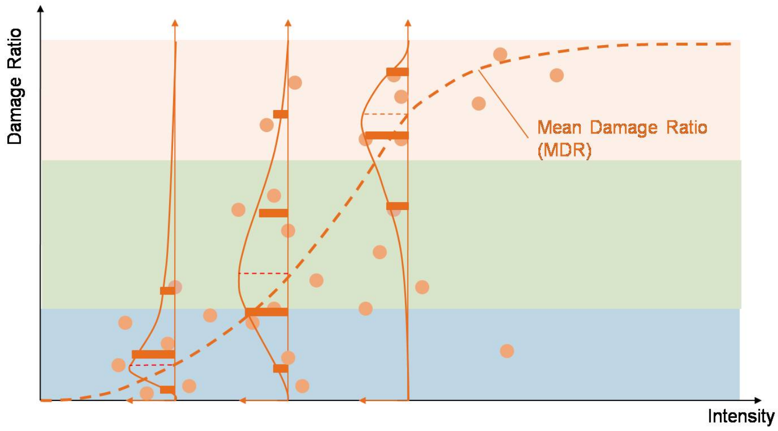

Figure 2: Different β-distributions for each mean value of intensity (horizontal axis)

(2.1.5) In reality, intensity is not a single point on the horizontal axis but each value of intensity has its own (known) distribution, where for $\log y$ distribution the mean is m and standard deviation is σ. So the question is how to combine a β-distribution of the damage around a particular mean intensity with the distribution of intensity itself? The answer to this question would give us a marginal distribution of damage D.

2.2 Mathematical solution: marginal distribution

(2.2.1) When the σ_{GMPE} of the ground motion is included into the damage assessment, the distribution of damage becomes a marginal distribution given by Bayes' theorem (previously the distribution of damage was given by $pdf_{D|\{Y=y\}}(t)$ in the integral below).

$$pdf_D(t) = \int pdf_Y(y) pdf_{D|\{Y=y\}}(t) dy. \quad (2)$$

(2.2.2) After filling in the distributions of the GMPE (given as lognormal distribution) and the β-distribution of damage for a given intensity, we get the expression:

$$pdf_D(t) = \int_{\mathbb{R}^+} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} t^{\alpha-1}(1-t)^{\beta-1} \frac{1}{y\sigma\sqrt{2\pi}} \exp\left(-\frac{(\log y - m)^2}{2\sigma^2}\right) dy, \tag{3}$$

where α and β are functions of $\mu(y)$.

(2.2.3) The above integral can be evaluated numerically; however, this method may be computationally- and time-intensive.

(2.2.4) Alternatively, an approximation of the intergral can be done using the Gaussian-Hermite Quadrature[4], such that we can re-arrange equation 3 to be the integral of an exponential distribution of the form below multiplied by a function, $f(x)$. This is approximated through a sum of polynomial functions.

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)dx \approx w_0 f(x_0) + \sum_{i=1}^{n} w_i(f(x_{i^+}) + f(x_{i^-})), \tag{4}$$

where $w_i = \frac{2^{n-1}n!\sqrt{\pi}}{n^2 H_{n-1}(x_i)^2}$ and x_i are roots of Hermite Polynomial. w_i and x_i can be easily evaluated to any order n by software packages such as Maple or Mathematica. The values are also available online.

(2.2.5) It can be shown that for odd $(2n+1)$, $x_0 = 0$ and $x_{i^-} = -x_{i^+}$.

(2.2.6) Let us approximate the integral 3 by 5 terms, so using equation 4, we obtain

$$pdf_D(t) \simeq W_2 pdf_{D|\{Y=y_2^-\}}(t) + W_1 pdf_{D|\{Y=y_1^-\}}(t) + \\ W_0 pdf_{D|\{Y=y_0\}}(t) + W_1 pdf_{D|\{Y=y_1^+\}}(t) + W_2 pdf_{D|\{Y=y_2^+\}}(t), \tag{5}$$

where

$$y_1^{\pm} = e^{\sqrt{2}\sigma x_{1^\pm}+m}, y_2^{\pm} = e^{\sqrt{2}\sigma x_{2^\pm}+m}, y_0 = \exp(m)$$

and we can calculate $W_0 = 0.533333375, W_1 = 0.22207585 W_2 = 0.01125728, x_{1^\pm} = \pm 0.958572, x_{2^\pm} = \pm 2.0201828$.

(2.2.7) The approximation can be done using 3 to 5 terms (or more depending on how many more steps can be accommodated without greatly increasing runtimes). A comparison of the 5-term approximation, the 3-term approximation, the full approximation and the original distribution of damage can be found in Figure 3.

[4]Please see reference [4] for proof and more information and also http://en.wikipedia.org/wiki/Gauss%E2%80%93Hermite_quadrature.

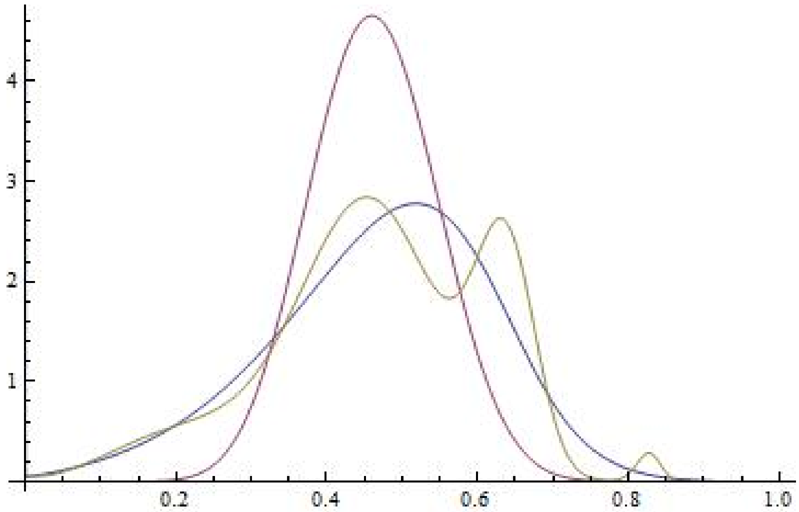

Figure 3: Comparison between a 1-term approximation - purple curve (current AIR's approach), full approximation calculated by Mathematica - blue curve, and a 5-term approximation - green curve.

(2.2.8) Since in practice a weighted average of several GMPEs is used, a weighted average of the probability distributions for each of the approximated GMPEs can be taken. This would ultimately give the overall distribution of damage resulting from the combinations of the distribution of each GMPE $pdf_Y(y)$ and the associated damage distributions given each GMPE $(pdf_{D|\{Y=y\}}(t))$.

2.3 Conclusions

(2.3.1) The Study Group proposed a way to incorporate the uncertainty associated with the intensity of ground motion to the calculations of damage. An analytical expression has been obtained for the marginal distribution of damage, expressed as an integral.

(2.3.2) The integral 3 has to be evaluated numerically. There may be many different approaches and the Study Group proposed to use the Gaussian-Hermite quadrature.

(2.3.3) The additional computational load will be 3 to 5 extra calculations which need to be saved (depending on the number of terms using to approximate the above integral). This is due to the requirement of saving the standard deviation of intensity and also the extra distributions in order to compute the marginal distribution of damage.

(2.3.4) The results obtained by evaluating the integral made physical sense and corresponded to the expectations for the marginal distribution of damage. For instance, the resulting distribution was wider reflecting more uncertainty contained in marginal distribution in comparison to the conditional distribution. Initial observations show that the mean of the marginal distribution of damage and the mean of the conditional distribution of damage are quite similar despite the inclusion of the distribution around intensity.

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[3] See for example, "Bayes' Theorem" article by James Joyce in the Stanford Encyclopedia of Philosophy, which provides a comprehensive introduction to Bayes' theorem. http://plato.stanford.edu/entries/bayes-theorem/

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