Electro-painting (P.P.G.)

1. Minor reformulation of the model*

The original formulation used the thickness of the deposited paint as a dependent variable. This is difficult to measure compared with the weight of deposited paint. Moreover the latter seems to be more directly related to the electrical resistance of the layer of paint.

Let $\mathbf{j}(\mathbf{x}, t)$ be the electrical current flowing in the paint bath, and let $\phi(\mathbf{x}, t)$ be the electric potential. Then the conservation of charge and Ohm's law give

$$\nabla \cdot \mathbf{j} = 0 \qquad \text{and} \qquad \mathbf{j} = -\sigma \nabla \phi$$

where σ is the electrical conductivity in the bath.

Let $w(\mathbf{x}, t)$ be the weight of the paint deposited per unit area of the metal surface, and let j_0 be the minimum current for paint to be deposited (representing an energy barrier either to flocculation or surface spreading). Then

$$\frac{\partial w}{\partial t} = \alpha [\mathbf{j} \cdot \mathbf{n} - j_0]_+ \qquad \text{and} \qquad \mathbf{j} \cdot \mathbf{n} = \frac{\phi}{R(w)} \qquad (1.1a,b)$$

where one can think of α as the weight of paint per unit electrical charge, and where $R(w)$ is the electrical resistance of the film of paint. The subscript $+$ means that paint is only deposited and is not allowed to dissolve off.

While there is some uncertainty about the correct relationship between resistance R and the weight w, we have taken a linear law for all the analysis below:

$$R = \beta w.$$

Figure 1 suggests that it may be more accurate to put $R = \tilde{\beta} w^{1.5}$. It is not yet known whether this result is statistically significant or relevant during most the paint deposition. Any law more complicated than a linear one would require a careful examination of the physical chemistry of the paint deposition. Early arriving paint globules may pack more or less efficiently, leaving smaller or larger gaps for salt solution to conduct. There may be some temperature history. A series of experiments would be required which carefully controlled all the variables, e.g. using a large heat sink to control temperature.

2. Typical magnitudes

The typical magnitudes of the variables involved are $\phi = 350\,\mathrm{V}$, $j = 1\,\mathrm{A\,m^{-2}}$, $j_0 = 0{\cdot}4\,\mathrm{A\,m^{-2}}$, $w = 4 \times 10^{-2}\,\mathrm{kg\,m^{-2}}$, $x = 1\,\mathrm{m}$, $t = 2 \times 10^2\,\mathrm{s}$, $\alpha = 4{\cdot}5 \times 10^{-5}\,\mathrm{kg\,A^{-1}\,s^{-1}}$, $\sigma = 10^{-1}\,\Omega^{-1}\,\mathrm{m}^{-1}$, $\beta = 10^4\,\Omega\,\mathrm{m}^4\,\mathrm{kg}^{-1}$, $\tilde{\beta} = 9 \times 10^4$ (SI units)

3. A one-dimensional problem

This is a simplified version of the problem of painting the outside of an object. Variations around the external surface will be ignored by assuming that there is just a simple linear variation in the potential from the anode to the surface of the paint film.

* A useful reference for sections 1-3: 'A model for an electropaint process', J.M. Aitchison, A.A. Lacey, M. Shillor, IMA J App. Math 33 (1984) 17-31

Let a potential $V(t)$ be applied at the anode, which is at a distance L from the paint surface. Let the potential at the paint surface be $\phi_s(t)$ and at the metal surface 0. Now the same current j will be flowing through the bath and the paint film, so Ohm's law in each gives

$$j = \sigma \frac{V - \phi_s}{L} = \frac{\phi_s}{\beta w}\,; \qquad \text{solving}, \; j = \frac{V}{\beta w + L/\sigma}$$

so

$$\frac{\partial w}{\partial t} = \alpha \left[\frac{V}{\beta w + L/\sigma} - j_0 \right]_+ \qquad (3.1)$$

Before solving this equation it is useful to note the scalings it gives. There is an initial current surge while the resistance across the paint film is comparable with the resistance across the bath, i.e. while the thickness is $w_i = L/\beta\sigma$ the current is $j_i = V\sigma/L$ over a time $t_i = w_i/\alpha j_i = L^2/\alpha\beta V\sigma^2$. With the typical magnitudes given above this gives a current surge of 35 $\mathrm{A\,m^{-2}}$ over a time of $\sim$1 s producing a paint deposit of $10^{-3}\,\mathrm{kg\,m^{-2}}$ which corresponds to about a micron thickness [the size of the paint globules?, for which the resistance law might not be well established?]. After the initial current surge the resistance is mostly across the paint film, so that the surface potential $\phi_s \sim V$.

The final weight of paint deposited is $w_\infty = V/\beta j_0$ and this takes a time $t_\infty \approx w_\infty/\alpha j_0$. With the above typical magnitudes this is a time of 3 hours to deposit $3.5 \times 10^{-2}\,\mathrm{kg\,m^{-2}}$.

After the initial current surge and before the final approach to the steady state, there is a simple intermediate time-scale behaviour governed by

$$\frac{\partial w}{\partial t} = \frac{\alpha V}{\beta w} \qquad \text{with solution} \qquad w = \sqrt{2\alpha V t}$$

The full equation can also be solved with little effort. Non-dimensionalising the weight with w_∞ and the time with t_∞, the equation reduces to

$$\frac{\partial w}{\partial t} = \left[\frac{1}{w + \gamma} - 1 \right]_+ \qquad \text{with solution} \qquad \ln \frac{1 - \gamma}{1 - w - \gamma} - w = t$$

in which $\gamma = w_i/w_\infty = Lj_0/V\sigma$ which is small. Figure 2 shows this prediction for the weight as a function of time compared with the measured deposition on a flat test plate. For short times (3 minutes) the linear and nonlinear resistance laws both give good agreement, but for longer times the nonlinear law is better.

4. Heat dissipation in the one-dimensional problem

The heat dissipated per unit area is $(V - \phi_s)j$ in the bath and $\phi_s j$ in the film. During the initial current surge, these are similar at $V^2\sigma/L$ which the typical magnitudes give as $12\,\mathrm{kW\,m^{-2}}$. This must cause considerable heating in the micron thin paint film. Once the electrical resistance across the film dominates that across the bath, the heat dissipation is concentrated in the film at $Vj(t) = V^2/\beta w(t)$. During the final approach to the steady state this will be Vj_0, which the typical magnitudes give as $0.2\,\mathrm{kW\,m^{-2}}$, which is still a lot of heat to dump into a thin paint film. It would be useful to look at heat conducted away into the metal surface, for which one needs the heat conductivities of the metal and the paint.

5. Optimal voltage ramp in the one-dimensional problem

The initial surge in current can be damaging (electrically and through heating the paint film – see above), so that the applied voltage V is sometimes ramped up to its final value, usually linearly. If the current is set to run at a limiting value j_{max} until the voltage reaches V_{max}, we have to find the time dependence for the applied voltage $V(t)$ and the time taken to reach V_{max}.

Now with $j = j_{max}$, the weight equation gives

$$w(t) = \alpha(j_{max} - j_0)t$$

and so the combined Ohm's laws in the bath and the film give

$$V(t) = j_{max}\left(\beta\alpha(j_{max} - j_0)t + L/\sigma\right)$$

i.e. a linear ramp from an initial small value of $j_{max}L/\sigma$. The time taken to reach V_{max} is $V_{max}/j_{max}^2\beta\alpha = t_\infty(j_0/j_{max})^2$. Thus if the current can run at 20 times the minimum current for deposition, it will take 20 s to reach the final voltage.

For a three dimensional car, the details of the optimal voltage ramp will depend on the exterior shape of the car, the shape of the bath and the positions of the anodes. The optimal ramp can however be expected to be roughly linear.

6. Time-dependent throw along a narrow channel

This is the problem of painting the interior of panels. Initially when there is bare metal at a potential 0 all along the channel, the electrical current into the wall will decay exponentially like $(4\sigma V/\ell)\exp(-\pi x/\ell)$ where ℓ is the channel width, x measured the distance along the channel and V is the potential in the bath at the entrance of the channel (equal to the anode voltage after the initial current surge in the exterior problem). Thus the current will exceed the minimum deposition value only in a small region near to the entrance. As paint is deposited here, however, it acts like a partial insulator and so the potential down the channel increases. Thus the paint can extend some distance from the entrance. Previous studies had found this important distance that the paint can be thrown as $\sqrt{V\ell\sigma/j_0}$. Here we examine the time-dependent growth and afterwards generalisations to narrow channels with arbitrary cross-sections.

Consider a long narrow channel between two flat metal plates with a separation ℓ. Let x measure the distance along the channel from an entrance at $x = 0$ where the potential V is applied. Let the paint be deposited in the region $0 \le x \le s(t)$. Using lubrication ideas for the narrow channel, we have that the potential is approximately (asymptotically) uniform across the width of the channel but will vary along it, so $\phi =$ $\phi(x,t)$. Thus the current flowing across each of the paint films on the two surfaces is $\mathbf{j}\cdot\mathbf{n} = \phi/\beta w(x,t)$. If the current flowing along the channel is $j(x,t)$, it will satisfy the conservation equation

$$\frac{\partial}{\partial x}(\ell j) = -\frac{2\phi}{\beta w}$$

(The factor 2 comes from the two surfaces, top and bottom.) Ohm's law in the channel gives $j = -\sigma\partial\phi/\partial x$. Hence the governing equation for the variation of the potential is

$$\frac{\partial^2\phi}{\partial x^2} = \frac{2}{\beta\sigma\ell}\frac{\phi}{w}$$

Given the distribution of paint at one instant $w(x,t)$, one solves this equation for ϕ, with boundary conditions

$$\phi = V \quad \text{at } x = 0 \qquad \text{and} \quad \phi = 0 = \frac{\partial \phi}{\partial x} \quad \text{at } x = s(t)$$

(One of the two conditions imposed at the end of the paint $x = s(t)$ should be thought as a regularity condition associated with $w \to 0$ as $x \to s(t)$.) With ϕ found, one can then integrate the paint equation

$$\frac{\partial w}{\partial t} = \alpha \left[\frac{\phi}{\beta w} - j_0 \right]_+$$

Before solving these equations it is useful to note the scalings they give. The final thickness of the paint will have a scale $w_\infty = V/\beta j_0$, which is the same as in the earlier one-dimensional (external) problem. While the paint has a uniform final thickness in the external problem, the paint thickness does vary along the channel in this internal problem. The time scale to throw the paint along the channel is $w_\infty/\alpha j_0 = V/\beta\alpha j_0^2$, which again is the same as in the earlier external problem. The scale for the distance the paint to be thrown along the channel comes from the ϕ-equation as $\sqrt{\beta\sigma\ell w_\infty/2} = \sqrt{V\sigma\ell/2j_0}$. The typical magnitudes give this as 0.35 m. Non-dimensionalising with the above scalings for the weight, time, x-lengths and with V for ϕ, the governing equations become

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\phi}{w} \qquad \text{and} \qquad \frac{\partial w}{\partial t} = \left[\frac{\phi}{w} - 1 \right]_+$$

7. The steady state

In the steady state $\partial w/\partial t = 0$ so $\phi/w = 1$, and so that the potential satisfies

$$\frac{\partial^2 \phi}{\partial x^2} = 1 \qquad \text{with } \phi = 1 \text{ at } x = 0 \qquad \text{and } \phi = 0 = \frac{\partial \phi}{\partial x} \text{ at } x = s$$

with solution
$$\phi = \tfrac{1}{2}(s - x)^2 \qquad \text{with } s = \sqrt{2}$$

Thus the dimensional throw is $\sqrt{V\sigma\ell/j_0}$, which may appear to be a factor of $1/\sqrt{2}$ smaller than results obtained before the 1990 study group, the difference being due to paint being deposited on both top and bottom surfaces in this analysis.

8. A similarity solution for short times

At short times the film is thin, while the potential is $O(1)$, and so $\phi/w \gg 1$ except near to the moving front at $x = s(t)$. Hence we look for a solution of $\partial w/\partial t = \phi/w$. Try a solution in the form

$$\phi = A(t)\big(s(t) - x\big)^n \qquad \text{and} \qquad w = B(t)\big(s(t) - x\big)^m$$

Substituting these forms into the potential equation gives $m = 2$ and $B = 1/n(n-1)$. The boundary conditions on the potential at the front $x = s(t)$ are then satisfied (although they are outside the region where $\phi/w \gg 1$). Substituting into the simplified weight equation gives $n = 3$ and $\dot{s} = A/mB^2$. Thus the similarity solution becomes

$$\phi = \frac{\dot{s}}{18}\big(s(t) - x\big)^3 \qquad \text{and} \qquad w = \tfrac{1}{6}\big(s(t) - x\big)^2$$

Finally applying the condition $\phi = 1$ at $x = 0$ yields an equation for the motion of the front

$$\dot{s} = 18/s^3 \qquad \text{so} \quad s = (72t)^{1/4}$$

The $t^{1/4}$ growth can be understood as first a $t^{1/2}$ growth of $w(0,t)$ where $\phi = 1$, with an x^2-dependence of $w(x,t)$ which vanishes at $s(t) = \sqrt{6w(0,t)}$.

Now some experiments seem to find not a $t^{1/4}$ behaviour, but instead a $t^{1/2}$ movement of the front. This can be explained by taking the voltage at the open end of the channel to increase linearly in time, i.e. $\phi = kt$ at $x = 0$ (in dimensional variables $k = \dot{V}t_\infty/V_{max}$). Then

$$\dot{s} = 18kt/s^3 \qquad \text{so} \quad s = (6t)^{1/2}k^{1/4}$$

9. 'Ad hoc' approximation for all times

Now the distribution of the paint has a quadratic dependence on distance down the channel in both the steady state and in the similarity solution above. It is no surprise therefore that the computer solutions in the following section also find a quadratic dependence for all times to a good approximation. Hence we assume

$$w(x,t) = w(0,t)\big(1 - x/s(t)\big)^2 \qquad \text{with} \qquad \frac{dw(0,t)}{dt} = \frac{1}{w} - 1$$

To obtain an equation for the movement of the front, it was noticed that the weight equation has a simple integral moment

$$\frac{d}{dt}\int_0^{s(t)} xw(x,t)\,dx \quad = \quad \int_0^s x\left(\frac{\partial^2\phi}{\partial x^2} - 1\right)dx \quad = \quad 1 - \tfrac{1}{2}s^2$$

This integral moment result is exact. If we now substitute the approximate quadratic form for $w(x,t)$, we obtain

$$\frac{d}{dt}\left(\tfrac{1}{12}w(0,t)s^2\right) \;=\; 1 - \tfrac{1}{2}s^2$$

Thus we have two ordinary differential equations for the unknowns $w(0,t)$ and $s(t)$.

10. Computer solutions

The simple numerical method adopted was to hold ϕ and w on a equally spaced finite difference grid on $0 \leq x \leq x_\infty$. At each time step the previous w was used in the instantaneous ϕ-equation, which was solved subject $\phi = 1$ at $x = 0$ and $\phi = 0$ at $x = x_{max}$ by a tridiagonal inversion. The resulting ϕ was then used in a simple forward time stepping of w.

To avoid problems with evaluating the expression ϕ/w when $w = 0$, a very small precursor film was given as the initial condition $w(x, 0) = w_0$. [The alternative would be to have a continuously deforming grid over just $0 \le x \le s(t)$, using $\dot{s} = -(\phi/w - 1)_x$ evaluated at $x = s(t)-$.] The small precursor reduces the final throw from $\sqrt{2}$ to $\sqrt{2 + w_0} - \sqrt{w_0}$, and thus quite small values of w_0 are required.

Because of very rapid initial activity in $w(0, t)$, very small time steps are required at the beginning, but are not needed later. Thus the time step was tied to $w(0, t)$ thorough $\delta t = w(0, t) \times tol$. Typical values of the numerical parameters were

$$x_\infty = 2, \quad \delta x = 0{\cdot}5 \times 10^{-2}, \quad w_0 = 10^{-3}, \quad tol = 10^{-3}$$

The motion of the painting front $s(t)$ is given in figure 3. It is seen that the front reaches 70% of its final position in a very short time of $t = 0{\cdot}05$ (corresponding to 10 minutes in the practical application), and then there is rather a slow climb reaching 90% by $t = 0{\cdot}3$. The initial rapid motion is seen to be roughly like $t^{1/4}$. The prediction of the similarity solution $s(t) = (72t)^{1/4}$ is good to $s = 0{\cdot}7$ at $t = 0{\cdot}005$. The 'ad hoc' approximation gives a reasonable prediction for all times.

Figure 4 shows numerical results for paint coming in from two open ends of the channel. The length of the channel L has been chosen to be 1·5, i.e. the distance either end can throw. The two ends have no effect on one another until $t = 0{\cdot}01$ when the two regions collide. Thereafter the current into the films, and hence the rate of depositing paint, becomes uniform along the channel. (Before the ends collide, the current into the films ϕ/w has a linear variation away from the ends of the channel, which is the variation of the similarity solution.) By $t = 0{\cdot}1$ the variation of the paint along the channel has adopted its equilibrium from

$$w(x, t) = w(0, t) - \tfrac{1}{2}x(L - x)$$

although the thickness at the ends $w(0, t)$ has only reached 40% of its final value.

11. Extension to axisymmetric spreading

The analysis above has been for a two-dimensional channel with no variation in the third direction. We now consider axisymmetric spreading between two flat plates with the paint bath being accessed through a circular hole in one of the plates. The radius of the hole r_{min} should be several times the separation of the plates ℓ in order for the long narrow channel approximation to be used. To study this axisymmetric spreading we need only modify the potential problem to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) = \frac{\phi}{w} \qquad \text{with } \phi = 1 \text{ at } r = r_{min} \qquad \text{and } \phi = 0 = \frac{\partial \phi}{\partial r} \text{ at } r = s$$

The same non-dimensionalisation applies to the axisymmetric spreading as to the two-dimensional spreading along a narrow channel.

The steady state has $\phi/w = 1$ so that the potential equation can be solved for

$$\phi = \tfrac{1}{2}s^2 \ln\frac{s}{r} - \tfrac{1}{4}s^2 + \tfrac{1}{4}r^2$$

Applying the condition $\phi = 1$ at $r = r_{min}$ yields an implicit result for the throw s

$$s = \sqrt{2} \Bigg/ \sqrt{\ln \frac{s}{r_{min}} - \tfrac{1}{2} + \tfrac{1}{2}\frac{r_{min}^2}{s^2}}$$

The distance thrown in axisymmetric geometry is therefore smaller by the square root in the denominator, but this varies very slowly. Thus the throw is reduced from 1·41 to 1·22 when $r_{min} = 0{\cdot}2$, to 0·91 when $r_{min} = 0{\cdot}05$, and to 0·71 when $r_{min} = 0{\cdot}008$.

Because the potential drops more rapidly away from the hole in the axisymmetric geometry compared to the two-dimensional problem, the steady weight of paint deposited will also drop off more rapidly.

The time-dependence of the axisymmetric spreading is plotted in figure 3. It shows a similar fast $t^{1/4}$ initial spreading, with half the eventual distance thrown by $t = 0{\cdot}005$.

12. Extension to narrow channels of arbitrary cross-sections

Consider spreading along a long narrow channel in the x-direction. Let the cross-section have an area $A(x)$ and a perimeter $C(x)$, which should vary only slowly along the channel. (The channel can also be slowly curved if x measures the distance along the curved channel.) The lubrication ideas which were used before for the long narrow geometry now yield a potential equation (in dimensional form)

$$\frac{\partial}{\partial x}\left(A(x)\frac{\partial \phi}{\partial x}\right) = \frac{C(x)}{\beta\sigma}\frac{\phi}{w}$$

Here is has been assumed that the paint is deposited evenly around the perimeter, which can be shown to be asymptotically correct except at occasional sharp corner in the cross-section (where is will be thicker).

In the steady state $\phi/\beta w = j_0$, so that integrating once from the front at $x = s$ where the current vanishes

$$A(x)\frac{\partial \phi}{\partial x} = -\frac{j_0}{\sigma}\int_x^s C(x')\,dx'$$

Integrating from $x = 0$ where $\phi = V$ then gives

$$\phi(x) = V - \frac{j_0}{\sigma}\int_0^x \frac{dx''}{A(x'')}\int_{x''}^s C(x')\,dx'$$

Finally applying $\phi = 0$ at $x = s$ gives a condition to determine the distance thrown s

$$\frac{V\sigma}{j_0} = \int_0^s \frac{dx''}{A(x'')}\int_{x''}^s C(x')\,dx' = \int_0^s \left(\int_0^{x'} \frac{dx''}{A(x'')}\right) C(x')\,dx',$$

the later form being easier for calculating the unknown s from given $A(x)$ and $C(x)$.

In the case of a cone tapering to zero at an apex at $x = x_c$,

$$A(x) = a(x_c - x)^2 \qquad \text{and} \qquad C(x) = c(x_c - x)$$

For these functions the shape integrals can be evaluated:

$$\int_0^{x'} \frac{dx''}{A(x'')} = \frac{1}{a}\left(\frac{1}{x_x - x'} - \frac{1}{x_c}\right) = \frac{x'}{ax_c(x_c - x')}$$

so $$\int_0^s \left(\int_0^{x'} \frac{dx''}{A(x'')}\right) C(x')\,dx' = \int_0^s \frac{cx'}{ax_c}\,dx' = \tfrac{1}{2}s^2\frac{c}{ax_c}$$

Hence the distance thrown is identical with the original two-dimensional calculation $\sqrt{V\sigma\ell/j_0}$ with the gap width of the channel ℓ being taken to be $2ax_c/C = 2A(0)/C(0)$, which would be the radius if the cross-section were circular.

13. Further problems

As recorded above, further experimental and then theoretical study is required on the formulation of the resistance law. This may be just a temperature effect, or some more complicated physical chemistry.

The study group did not address the production of a three-dimensional time-dependent computer code for the external problem or for internal problems which were not narrow channels. For handling the real complex geometries, the boundary element method has much to commend it, although it was argued that finite-differences could be faster.

Contributions from
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Figures

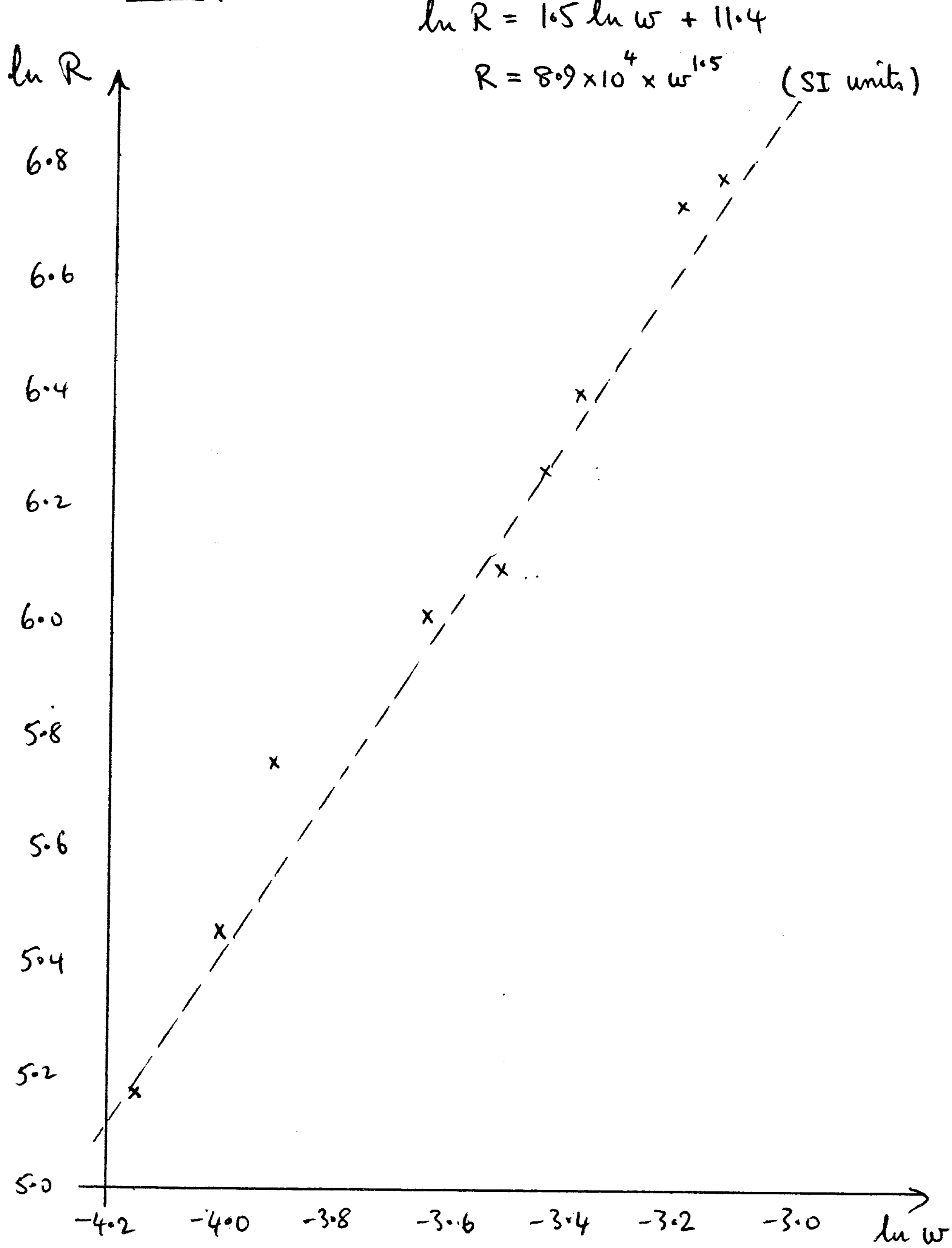
Figure 1. The resistance of the paint film as a function of the weight deposited.

Figure 2. The predicted weight deposited as a function of time compared to the experimentally observed deposition.

Figure 3. The time-dependent throw of paint along narrow channel.

Figure 4. The rate of depositing paint as a function of position along a narrow channel with two ends open to the bath. The different curves are for different times.

Fig 1.

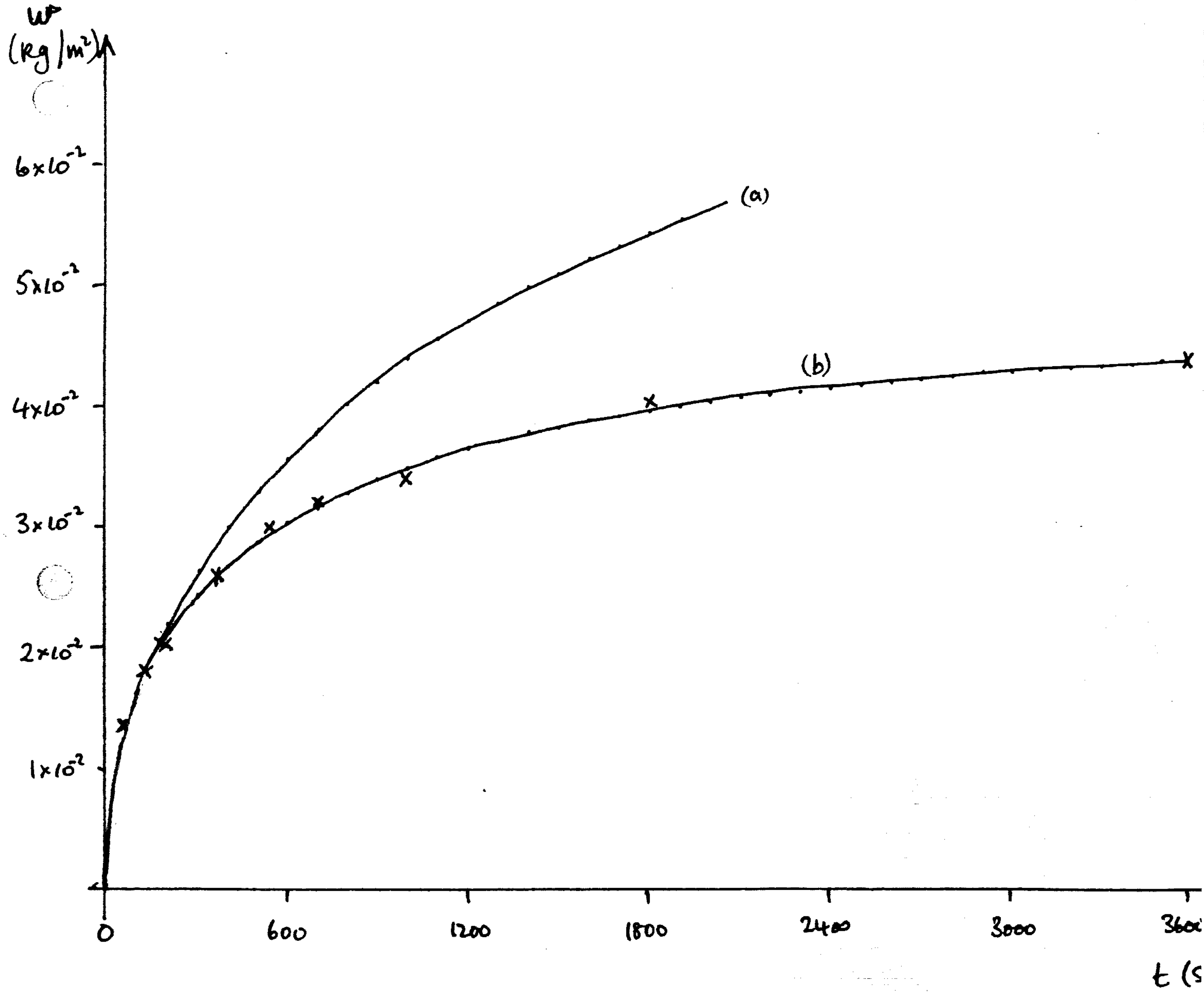


R is calculated from data supplied for the current density and from (1.16) with $\phi = 350V$ (assuming that almost all the potential drop is across the film).

Fig2 One-dimensional growth : w versus t. The crosses are experimental results. Curve (a) is the solution of (3·1) with $\beta = 10^4$; curve (b) is a numerical solution of the equation corresponding to (3·1) when $R = \tilde{\beta} w^{1.5}$:

$$\frac{\partial w}{\partial t} = \alpha \left[\frac{V}{\tilde{\beta} w^{1.5} + L/\sigma} - j_0 \right].$$

$\alpha = 4.5 \times 10^5$, $V = 350$, $L = 1$, $\sigma = 0.1$, $j_0 = 10^4$, $\tilde{\beta} = 8.9 \times 10^4$.



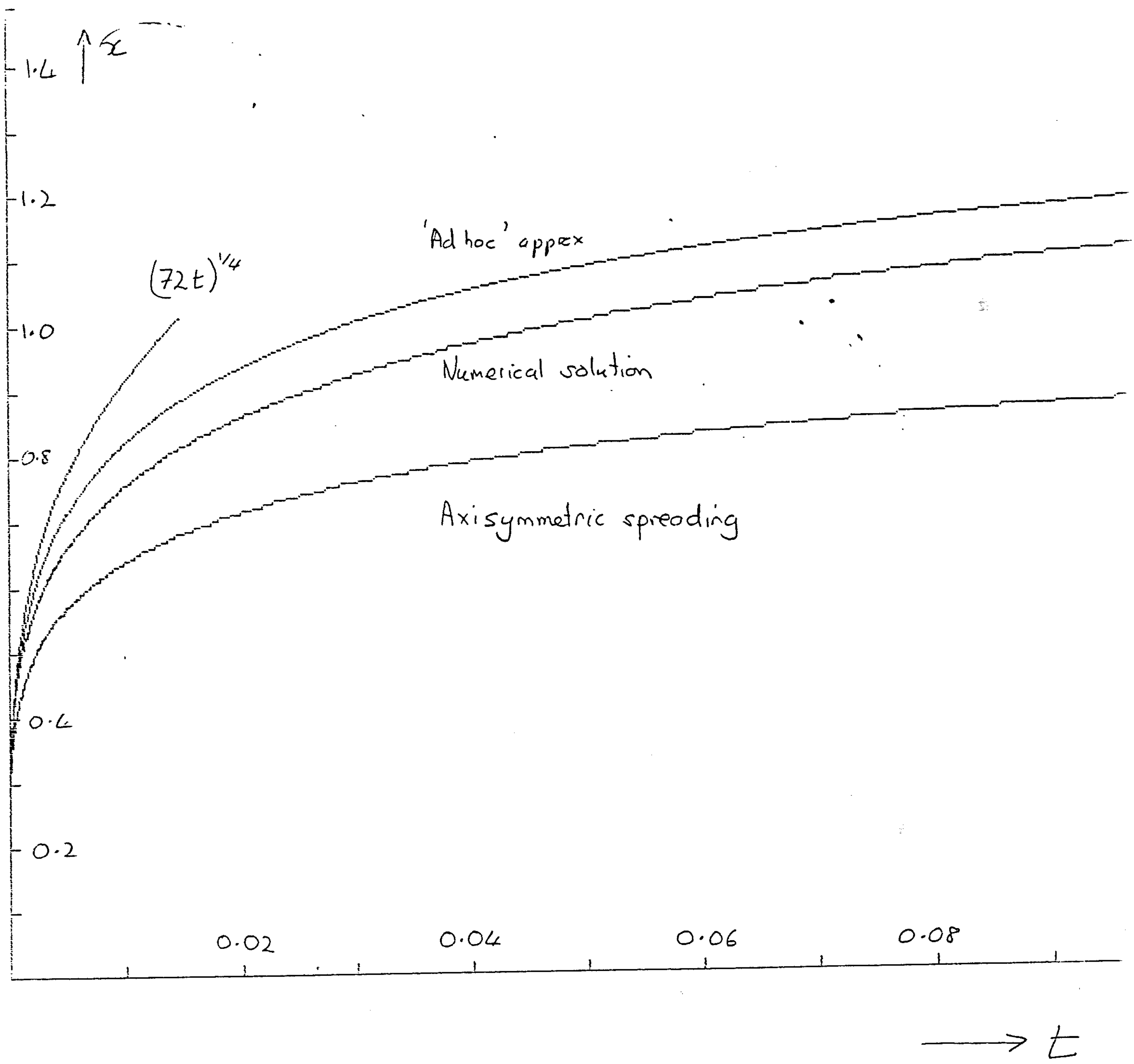


Figure 3. The time-dependent throw along a narrow channel.

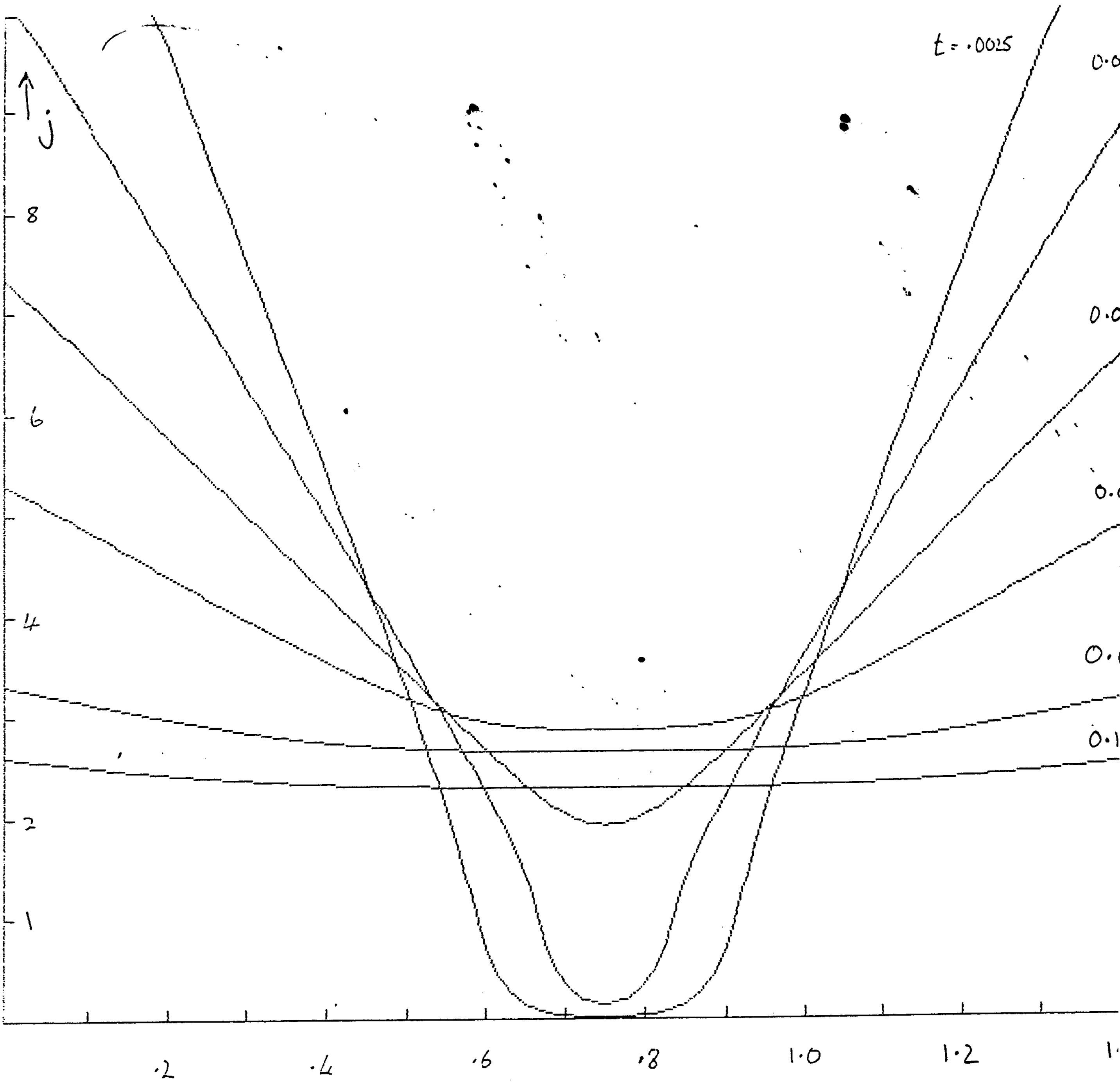


Figure 4. The normal current into the wall $j = \varphi / w$ as a function of position x along a narrow channel with tw[o] open ends at $x = 0$ and at $x = 1{\cdot}5$. The different curves ar[e] for different times.

Postscript:

1. The 'roof' problem. One can analyse a general thin layer (e.g. between a rectangular roof and the bath surface), The analysis is a simple extension of Section 6 (or 11) and give the (dimensional) equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{j_0}{\sigma \ell} \qquad (1)$$

where ℓ is the thickness (there is no factor 2 because there is no paint deposited at the top). This to be solved with $\phi = V$ at the edges. The problem where there are bare patches is the well-known 'obstacle problem' but this is presumably not of interest. Without bare patches, (1) can be solved for example by a series expansion. For a rectangle $-a < x < a$, $-b < y < b$, with $\phi = V$ on the edges,

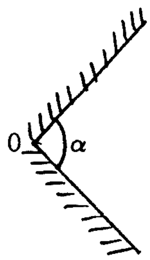
$$\phi = V + \frac{j_0}{\sigma \ell}\left[(y^2-b^2)/2 + \Sigma(-1)^n a_n \cos(\lambda_n y)\cosh(\lambda_n x)\right]$$

where $\lambda_n = (2n+1)\pi/2b$ and $a_n = (-1)^n/b\lambda^3{}_n \cosh(\lambda_n a)$. The smallest value of ϕ (and hence w) is at $x = y = 0$. A good approximation to this value is obtained by taking only the $n = 0$ term from the expansion; this yields $\phi(0,0) = V - \frac{b^2 j_0}{\sigma \ell}(\frac{1}{2}-8/\pi^3\cosh(\pi a/2b))$. For a square, $a = b$ and $\phi(0,0) \approx V - 0.41 b^2 j_0/\sigma\ell$.

Simpler still is a circular roof of radius b; here $\phi = V - j_0(b^2-x^2-y^2)/4\sigma\ell$ with a minimum value $V - b^2 j_0/4\sigma\ell$.

2. Corners

The (possibly important) problem of painting an interior corner was not discussed at the Study Group. Consider a corner of angle α as shown:



With $j_0 = 0$, and using the linear resistance law, the local similarity solution is of the form

$$\phi = t^{k/(2-k)}\Phi(rt^{-1/(2-k)}), \quad w = t^{1/(2-k)}W(rt^{-1/(2-k)}),$$

where $k = \pi/\alpha < 2$. Thus $w(0,t) = AL^{-k/(2-k)}t^{1/(2-k)}$ where A is a geometrical factor and L a typical length scale (distance to the anode). This suggests that as $\alpha \to \pi/2$ from above, painting becomes slower and slower, while for $\alpha < \pi/2$ it may not occur at all for a finite time (this is the case for $\alpha = 0$, the parallel-sided box). Further work may be possible here, including the incorporation of a non-zero radius of curvature at 0.

3. The 'long thin' equations for w and ϕ in a channel have been analysed by Anwar Meirmanov, who has proved existence and uniqueness of a solution; we hope for a manuscript in due course....