Model Order Reduction for Electronic Circuits: Mathematical and Physical Approaches

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1 Introduction: Electronic Circuit Simulation

Electronic circuits are ubiquitous; they are used in numerous industries including: the semiconductor, communication, robotics, auto, and music industries (among many others). As products become more and more complicated, their electronic circuits also grow in size and complexity [1]. This increased the need for circuit simulators to evaluate potential designs before fabrication, as integrated circuit prototypes are expensive to build, and troubleshooting is difficult. Circuit simulators were first created in the late 1960's and become more popular in the 1970's due to the explosive growth of the integrated circuit market [2].

In this report, we focus on the simulation of printed circuit boards (PCB's) and interconnects (as illustrated in Figure 1) both of which are of great importance in the semiconductor industry. Interconnect and PCB simulation proceeds by using Maxwell's equations to create a mathematical model. The boundary element method is then used to discretize the equations, and the variational form of the equations are then solved on the graph network (an example of which is shown in Figure 1). This can then be co-simulated with the circuit on the PCB or the circuit below the interconnect structure

The graph network used for the circuit simulation is very large and cumbersome since it is automatically generated via a commercial software package. Because the graph network

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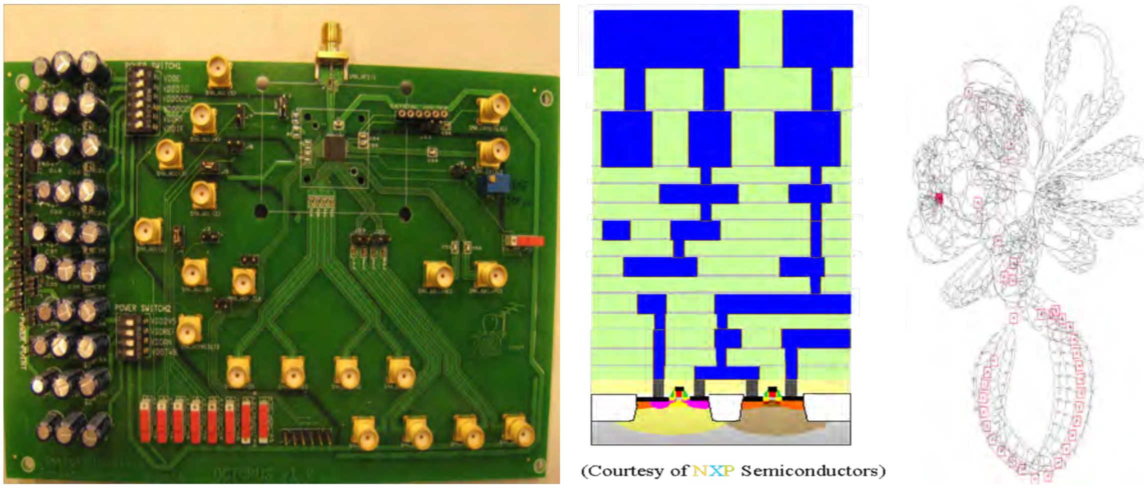


Figure 1 The electronic application: a printed circuit board diagram (left), an interconnect structure (middle), and a simple graph network (right).

must be connected to the circuit simulator, a reduced order model, which will allow for realistic simulations, is needed.

In this report, we focus on the development of mathematical and physical model order reduction techniques and corresponding theory for the simulation of PCB's and interconnects. Two problems were formulated in this context. The goal of the first problem is to develop a mathematical algorithm to reduce the size of the graph network which models the substrate of the silicon layer in a PCB as a purely resistive network. This is discussed in Sections 2 and 3. The goal of the second problem is to develop physically-realisable reduced order models for the interconnect network above the silicon layer which cannot be modeled as a purely resistive network. We introduce this question in Section 4. Stability, passivity, and realisability are important issues in the creation of physically-realisable reduced order models, and are addressed in Sections 5, 6, and 7. Finally, in Section 8, we briefly summarize the main results of the study group report.

2 Model Order Reduction in the Purely Resistive Case

To present the system of equations describing the physical properties of the circuit, we need the following definitions.

Definition 2.1 A *directed graph* or *network* is a couple $G = (U, \mathcal{A})$, where U is a finite set and $\mathcal{A}$ a subset of $U \times U$. The members of U are called *nodes* and those of $\mathcal{A}$ *edges*. G is connected if for any pair u, v of nodes, there exists an (undirected) path from u to v in G.

An edge of G will be denoted (u, v) or uv for short. The order of the vertices matters because the graph is directed. As usual in graph theory, n will denote the number of nodes of G and m the number of edges of G.

Definition 2.2 Let $G = (U, \mathcal{A})$ be a directed graph. We say that G is *complete* if it does not contain (u, u) for any $u \in U$ and contains exactly one of (u, v) and (v, u) for any pair $\{u, v\}$ of vertices.

We also need some preliminaries concerning the modelling of interconnect structure, substrates, and circuits. For a structure modelled with a network consisting of resistors only, the system of Kirchhoff's equations is

$$R\,I - P\,V = 0,\ P^T\,I = J,$$

where R is an $m \times m$ diagonal resistance matrix, P the edge-node incidence matrix of G (of dimension $m \times n$), I an m-dimensional vector of currents flowing in the edges, V an n-dimensional vector of voltages at the nodes, and J an n-dimensional vector of terminal currents flowing into the interconnect system. J usually has many zeros, and is nonzero only at the "external" nodes. Consequently, we are often only interested in M^{-1} restricted to the terminals. From the above linear system one derives the equation $\left(P^T R^{-1} P\right) V = J$. Let M denote the matrix $P^T R^{-1} P$. From M^{-1} one can deduce the resistances between all pairs of nodes.

A node of the circuit is either an *external node* or an *internal node*; accordingly, we can partition the node set U into the sets U_e (the subset of external nodes) and U_i (the set of internal nodes). In order to test the circuit, it suffices to consider external nodes and the resistances between pairs of such nodes. The first idea that comes to mind is to replace the graph G by a complete graph G' on U_e in which the resistance between two nodes u and v is the same as that deduced from M^{-1} (for any pair $\{u, v\}$). Of course, this requires the computation of M^{-1}. Unfortunately, this computation is expensive since the number of internal nodes is in general much larger than the number of external nodes. Problem 1 is to replace the graph G by a graph G' containing all the external nodes but fewer internal nodes, and the matrix $M = P^T R^{-1} P$ by a matrix $M' = (P')^T (R')^{-1} P'$, where P' is the edge-node incidence matrix of G' and R' is relatively easy to compute. In the next section we show how this can be achieved if G contains a certain type of cutset.

2.1 Model reduction by means of cutsets. We need a few more definitions from graph theory in order to describe a broad class of reductions.

Definition 2.3 Let $G = (U, \mathcal{A})$ be a directed graph and U' a subset of U. We let $\mathcal{A}(U')$ denote the set of edges both ends of which belong to U'. The subgraph induced by U' is the graph $(U', \mathcal{A}(U'))$.

Definition 2.4 Let $G = (U, \mathcal{A})$ be a connected network. A *cutset* U' is a subset of U such that the removal of U' results in a disconnected network.

Definition 2.5 Let $G = (U, \mathcal{A})$ be a network. Given a nonempty and proper subset U' of U, we define the *undirected cut* which shores U' and $\overline{U'}$ as the set of all edges uv such that either $u \in U'$ and $v \in \overline{U'}$ or $u \in \overline{U'}$ and $v \in U'$ holds.

In what follows we shall use the word "cut" to mean "undirected cut".

We will now try to formalize the application of the idea of "cutset" to the problem of reducing circuit size. Consider again the equation $\left(P^T R^{-1} P\right) V = J$ and assume that in the underlying network (which is purely resistive), there is a cutset U_2. The removal of U_2 creates two subnetworks whose vertex sets are denoted by U_1 and U_3, respectively. We assume that U_1 does not contain any external node.

The matrix P can thus be partitioned as

$$\begin{pmatrix} A & 0 & 0 \\ B & C & 0 \\ 0 & D & 0 \\ 0 & E & F \\ 0 & 0 & G \end{pmatrix},$$

where A (resp. D, G) is the edge-node incidence matrix of the subnetwork induced by U_1 (resp. U_2, U_3), the submatrix $[B, C]$ is the edge-node incidence matrix of the cut with shores U_1 and $\overline{U_1}$, and the submatrix $[E, F]$ is the edge-node incidence matrix of the cut with shores $\overline{U_3}$ and U_3.

Example 2.6 In this example, U_1 is the set $\{1, 2, 3, 4, 5, 6, 7\}$, U_2 the set $\{8, 9, 10, 11\}$, and U_3 the set of remaining nodes. Nodes 8 and 11 are external nodes, but U_1 does not contain any external node (although U_3 may contain some). Note that the subgraph $(U_2, E(U_2))$ may have an arbitrary "shape"; in the example, this subgraph is a path, but it could also be a cycle, and it need not even be connected.

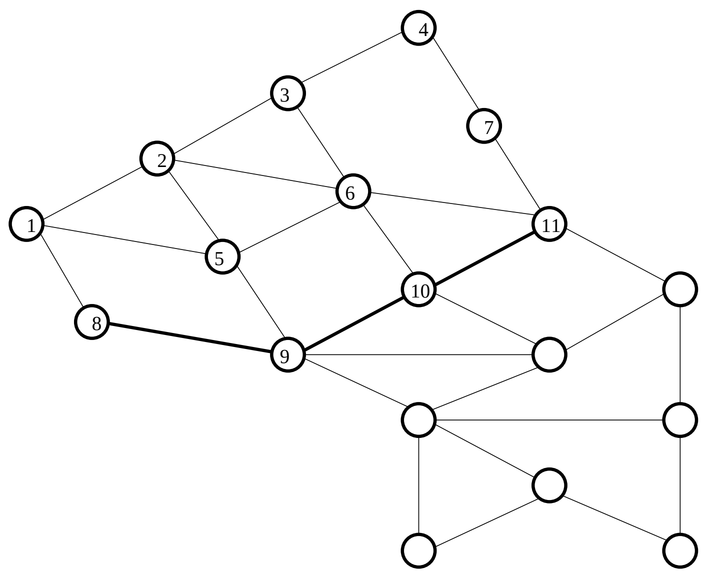


Figure 2 A network with a cutset

In general, the matrix $P^T R^{-1} P$ is the following.

$$\begin{pmatrix} A^T & B^T & 0 & 0 & 0 \\ 0 & C^T & D^T & E^T & 0 \\ 0 & 0 & 0 & F^T & G^T \end{pmatrix} \begin{pmatrix} R_1^{-1} & 0 & 0 & 0 & 0 \\ 0 & R_2^{-1} & 0 & 0 & 0 \\ 0 & 0 & R_3^{-1} & 0 & 0 \\ 0 & 0 & 0 & R_4^{-1} & 0 \\ 0 & 0 & 0 & 0 & R_5^{-1} \end{pmatrix} \begin{pmatrix} A & 0 & 0 \\ B & C & 0 \\ 0 & D & 0 \\ 0 & E & F \\ 0 & 0 & G \end{pmatrix} =$$

$$\begin{pmatrix} A^T R_1^{-1} A + B^T R_2^{-1} B & B^T R_2^{-1} C & 0 \\ C^T R_2^{-1} B & C^T R_2^{-1} C + D^T R_3^{-1} D + E^T R_4^{-1} E & E^T R_4^{-1} F \\ 0 & F^T R_4^{-1} E & F^T R_4^{-1} F + G^T R_5^{-1} G \end{pmatrix}.$$

Let the matrix on the right-hand side of the above equation be denoted by $M_{substrate}$.

Finally, the system $\left(P^T R^{-1} P\right) V = J$ can be rewritten as

$$M_{substrate} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} \tag{2.1}$$

for some vectors x, y, z, r_1, r_2 and r_3 of appropriate dimensions. In this context, "reducing the model" means removing the nodes in U_1 and replacing the subgraph $(U_2, \mathcal{A}(U_2))$ by a subgraph $(U_2, \mathcal{A}_0)$ (whose edge-node incidence matrix is H) such that

$$\begin{pmatrix} H^T R_6^{-1} H + E^T R_4^{-1} E & E^T R_4^{-1} F \\ F^T R_4^{-1} E & F^T R_4^{-1} F + G^T R_5^{-1} G \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} r_2 - \left(-C^T R_2^{-1} B A'\right) r_1 \\ r_3 \end{pmatrix} \tag{2.2}$$

has the same solution as Equation (2.1).

Let us denote by A' the inverse of the upper left submatrix, i.e., the inverse of $A^T R_1^{-1} A + B^T R_2^{-1} B$. Then x equals $A'\left(r_1 - B^T R_2^{-1} C y\right)$, and the replacement of x in the middle part of the system yields

$$\left(C^T R_2^{-1} B\right) A' \left(r_1 - B^T R_2^{-1} C y\right) + \left(C^T R_2^{-1} C + D^T R_3^{-1} D + E^T R_4^{-1} E\right) y + E^T R_4^{-1} F z = r_2,$$

and thus

$$\left(-C^T R_2^{-1} B A' B^T R_2^{-1} C + C^T R_2^{-1} C + D^T R_3^{-1} D\right) y + E^T R_4^{-1} E y + E^T R_4^{-1} F z = r_2 - \left(C^T R_2^{-1} B A'\right) r_1.$$

Let H be the incidence matrix of any complete graph on the set U_2. One wishes to compute a matrix R_6 of resistances such that

$$-C^T R_2^{-1} B A' B^T R_2^{-1} C + C^T R_2^{-1} C + D^T R_3^{-1} D = H^T R_6^{-1} H.$$

The system $\left(P^T R^{-1} P\right) V = J$ could then be replaced by the system given in Equation (2.2) above. Let us now look at the matrix $H^T R_6^{-1} H$. Its rows and its columns correspond to the vertices in U_2. The entry of the matrix at the intersection of row u and column v equals $-\frac{1}{r}$, where r is the new resistance of edge (u, v). The sum of the columns of H is the column-vector consisting of 0's (denoted $\mathbf{0}$), and this implies that $\left(H^T R_6^{-1} H\right) \mathbf{1}$ equals $\mathbf{0}$ as well (where $\mathbf{1}$ denotes the vector all of whose components equal 1). We now show that the matrix

$$-C^T R_2^{-1} B A' B^T R_2^{-1} C + C^T R_2^{-1} C + D^T R_3^{-1} D$$

has the same property. We also have to show that A' exists and every off-diagonal entry of $C^T R_2^{-1} B A' B^T R_2^{-1} C$ is nonnegative.

Lemma 2.7 *The matrix $A^T R_1^{-1} A + B^T R_2^{-1} B$ is invertible and every off-diagonal entry of $C^T R_2^{-1} B A' B^T R_2^{-1} C$ is nonnegative.*

Proof For any nonzero vector z we have $z^T A^T R_1^{-1} A z \geq 0$ and $z^T B^T R_2^{-1} B z > 0$. Therefore $A^T R_1^{-1} A + B^T R_2^{-1} B$ is positive definite and hence invertible. On the other hand, its off-diagonal entries are nonpositive, i.e., $A^T R_1^{-1} A + B^T R_2^{-1} B$ is a Z-matrix. Since its eigenvalues are strictly positive, $A^T R_1^{-1} A + B^T R_2^{-1} B$ is also an M-matrix, which implies that it is inverse-positive (see [3] for the definitions of Z-matrix and M-matrix and a proof of this fact). This means that the entries of A', the inverse of $A^T R_1^{-1} A + B^T R_2^{-1} B$, are all nonnegative. Finally, all entries of the matrix $C^T R_2^{-1} B A' B^T R_2^{-1} C = (B^T R_2^{-1} C)^T A' B^T R_2^{-1} C$

are nonnegative because the entries of A' are nonnegative and those of $B^T R_2^{-1} C$ are nonpositive. $\square$

It follows easily from the preceding lemma that the off-diagonal entries of the matrix

$$-C^T R_2^{-1} B A' B^T R_2^{-1} C + C^T R_2^{-1} C + D^T R_3^{-1} D$$

are nonpositive, since $C^T R_2^{-1} C$ is a diagonal matrix and $D^T R_3^{-1} D$ a Z-matrix. This implies that the off-diagonal entries of $H^T R_6^{-1} H$ "make sense" from a physical point of view (i.e., the new resistances are nonnegative quantities).

Lemma 2.8 *The column sum of the matrix*

$$-C^T R_2^{-1} B A' B^T R_2^{-1} C + C^T R_2^{-1} C + D^T R_3^{-1} D$$

equals $\mathbf{0}$*, i.e.,*

$$\left(-C^T R_2^{-1} B A' B^T R_2^{-1} C + C^T R_2^{-1} C + D^T R_3^{-1} D\right) \mathbf{1} = \mathbf{0}$$

holds.

Proof We first show that $\left(A' B^T R_2^{-1} C\right) \mathbf{1}$ equals $-\mathbf{1}$. This is equivalent to proving that $\left(B^T R_2^{-1} C\right) \mathbf{1}$ equals $-\left(A^T R_1^{-1} A + B^T R_2^{-1} B\right) \mathbf{1}$. But we have

$$-\left(A^T R_1^{-1} A + B^T R_2^{-1} B\right) \mathbf{1} = -\left(B^T R_2^{-1} B\right) \mathbf{1} = \left(B^T R_2^{-1} C\right) \mathbf{1},$$

where the first equality follows from the fact that the column sum of A equals $\mathbf{0}$ and the second from the fact that $-B\,\mathbf{1}$ equals $C\,\mathbf{1}$.

Since $\left(A' B^T R_2^{-1} C\right) \mathbf{1}$ equals $-\mathbf{1}$ and $\left(D^T R_3^{-1} D\right) \mathbf{1}$ equals $\mathbf{0}$, we obtain

$$\left(-C^T R_2^{-1} B A' B^T R_2^{-1} C + C^T R_2^{-1} C + D^T R_3^{-1} D\right) \mathbf{1} = \left(-C^T R_2^{-1} B A' B^T R_2^{-1} C + C^T R_2^{-1} C\right) \mathbf{1}$$
$$= -C^T R_2^{-1} B A' B^T R_2^{-1} C\,\mathbf{1} + C^T R_2^{-1} C\,\mathbf{1} = -C^T R_2^{-1} B\,(-\mathbf{1}) + C^T R_2^{-1} C\,\mathbf{1} = \mathbf{0}.$$

$\square$

It follows from this lemma that if the entry of $H^T R_6^{-1} H$ at the intersection of row u and column v (for any u, v such that $u \neq v$) is equal to the corresponding entry of the matrix $-C^t R_2^{-1} B A' B^t R_2^{-1} C + C^t R_2^{-1} C + D^t R_3^{-1} D$, then the two matrices are identical.

We can now sketch a high-level algorithm for reducing the size of the original system.

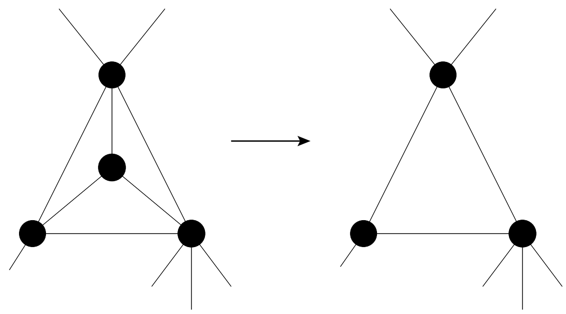
1. Find a cutset U_2 such that one shore of U_2 contains all the external nodes.
2. Compute the inverse of $A^T R_1^{-1} A + B^T R_2^{-1} B$, denoted A'.
3. Compute the matrix $-C^T R_2^{-1} B A' B^T R_2^{-1} C + C^T R_2^{-1} C + D^T R_3^{-1} D$.
4. Remove the nodes in U_1 and the edges incident to them from the circuit; replace the subgraph induced by U_2 by the graph H; and assign resistances to the edges of H using the information computed in Step 3.

Not all such reductions will be useful, however. Let n_1 (resp. n_2) denote $|U_1|$ (resp $|U_2|$). The cutset U_2 must be such that n_1 is relatively small when compared to n, the total number of nodes; otherwise computing the inverse of $A^T R_1^{-1} A + B^T R_2^{-1} B$ could be almost as expensive as computing the inverse of $P^T R^{-1} P$. On the other hand, U_2 cannot contain many nodes (i.e., n_2 must not be too large); otherwise the resulting circuit will be dense and its testing will consume too much time. This means we need to do a check as to whether or not the "reduction" is useful. For example, eliminating all internal nodes is usually not an option.

3 Graph Theoretic Algorithm

In this section, we further specify the reduction of the graph network outlined in Steps 1 and 4 above. Our algorithm for model reduction is based on the following rules:

1) The reduced graph must contain all external nodes of the original graph
2) The following figure illustrates the replacement rule we use.



This is known as the star-delta transformation in electronics and is related to Gaussian Elimination. This transformation is valid, because the resistances on the edges of the Δ can be modified in such a way so as to maintain the resistances between each pair of external nodes. Our strategy is to eliminate the regions that do not contain external nodes by finding the shortest paths between all such pairs. Afterwards, we remove the edges that have only one endpoint belonging to the shortest paths. We construct a complete graph on the remaining edges as motivated by the above replacement rule.

Our algorithm is then as follows:

Step 1: Determine the shortest path between all external nodes $E_1, \ldots, E_n$.
Step 2: List all edges $e_1, \ldots, e_j$ which have at least one end point belonging to any shortest path.
Step 3: Remove all edges stated in Step 2.
Step 4: Identify all components $C_k, k = 1, \ldots, t$ in this new graph.
Step 5: Apply the replacement rule if the number of edges in the complete graph $\tilde{K}_i$ is less than or equal to the number of edges in C_k.
Step 6: Replace the edges from Step 2 for the components that did not replaced with a complete graph.

Remark. One possible way to improve the algorithm's efficiency might be to construct only an expander graph of the remaining nodes (after constructing the shortest path between all pairs of external nodes), because constructing a complete graph would significantly increase the number of edges, and may thus not be very useful from a computational perspective. An expander graph has high connectivity but is sparse. We believe that the insertion of an expander graph would not violate the replacement rule.

In an extension of this report, the group is working to implement our graph theoretic algorithm and to obtain results on NXP graph networks.

4 Model Order Reduction for Physically Realisable Electronic Circuit Design

We now turn our attention to the interconnect network above the silicon layer on the chip. This network provides the necessary connections between the individual semiconductor devices in the silicon. As a rule of thumb, the larger the distance between two devices, the higher will be the metal layer in which the connection takes place. (See Figure 1 for an example of an interconnect structure; interconnect structures can have up to ten metal layers.) Problem 2 is to determine physically-realisable reduced order models for this interconnect network; this involves proving that the reduced system is stable and passive and can be constructed from resistors, inductors, and capacitors.

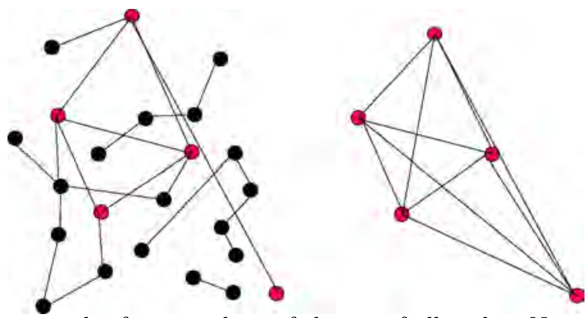


Figure 3 The super nodes form a subset of the set of all nodes. Note that they are not necessarily all exterior nodes.

The behaviour of the interconnect network of metal tracks can be determined only by solving the Maxwell equations. The component equations (Maxwell and Kirchoff laws) for the entire fine mesh circuit discretized via the boundary element method leads to the system of equations

$$\begin{aligned} (R+sL)\,I - PV &= 0, \\ P^T + sCV &= J \qquad (s = j\omega). \end{aligned} \tag{4.1}$$

Here we are looking at the system under the action of a source function proportional to e^{st} where s is a complex constant, usually taken to be $s = j\omega$, where ω is a frequency and j the imaginary unit. The use of a Laplace transform changes the system of differential equations describing the circuit into a system of algebraic equations that can be rewritten as follows:

$$\left[\begin{pmatrix} R & -P \\ P^T & 0 \end{pmatrix} + j\omega \begin{pmatrix} L & 0 \\ 0 & C \end{pmatrix}\right]\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ J \end{pmatrix}. \tag{4.2}$$

We now wish to focus on a subset of nodes, the so called *super nodes* (see Figure 3). The nodes represent a coarse graining of the full set of nodes. The coarse graining is associated with a natural distance imposed by the maximum frequency ω applied to the system. It makes little sense to have nodes on a finer scale than the wavelength of the electromagnetic wave of frequency $\nu = \omega/2\pi$. In this sense, a frequency ω corresponds to a length scale d given by

$$d \approx \frac{2\pi c}{\omega}$$

Stability can be mathematically phrased in terms of location in the complex plane of roots of the characteristic equation for the system matrix. We turn now to the question of what happens to the system matrix when we go algorithmically from the set of all nodes to the set of super nodes.

5 Stability of the Super Node Algorithm

The objective is to determine whether or not the matrix of the reduced system (after the non-super nodes have been eliminated) is stable, given that R, L, and C are symmetric positive definite (SPD).

The stability of the reduced system may be established by determining if the original system, now in the form (4.2), has all its eigenvalues in the right half-plane for any value of the physical parameter s.

Theorem 5.1 *The eigenvalues of the reduced system all lie in the right half-plane. Thus, the reduced system is stable for any value of the physical parameter s.*

Proof We consider two cases, i.e., $s = 0$ and $s \neq 0$.

Case 1: $s = 0$**.** If $s = 0$, system (4.2) becomes

$$\begin{bmatrix} R & -P \\ P^T & 0 \end{bmatrix} \begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ J \end{pmatrix}, \tag{5.1}$$

which can be reduced by elimination to the following,

$$P^T R^{-1} P V = J. \tag{5.2}$$

The coefficient matrix $P^T R^{-1} P$ is called the admittance matrix, and it is easy to show that it is SPD. (See also Section 2.) Hence, the reduced system (5.1) of super nodes is also SPD being a principal submatrix of the admittance matrix. Therefore, the eigenvalues for the reduced system (5.1) lie in the right half-plane.

Case 2: $s \neq 0$**.** In this case, $\omega \neq 0$, and the matrix in system (4.2) can be expressed in the form

$$A + jB, \tag{5.3}$$

where

$$A = \begin{pmatrix} R & -P \\ P^T & 0 \end{pmatrix}, \tag{5.4}$$

and

$$B = \omega \begin{pmatrix} L & 0 \\ 0 & C \end{pmatrix}. \tag{5.5}$$

We will use contradiction to prove that all the eigenvalues of this complex matrix, consisting of the real matrices A and B as the real and imaginary parts, lie in the right half-plane.

Suppose that there is an eigenvalue, ϵ, in the left half-plane. Then,

$$\det\left(A + jB - \epsilon I\right) = \det\left(\begin{bmatrix} R + j\omega L & -P \\ P^T & j\omega C \end{bmatrix} - \epsilon I \right) = 0. \tag{5.6}$$

There is continuous dependence of the eigenvalues on the parameter. So, by continuity, there exists an ω for which ϵ lies on the imaginary axis. That is, $\epsilon = -\epsilon_0 j$. Using a Schur complement approach,

$$\det(A + jB + \epsilon_0 jI) = \det(j\omega C - \epsilon_0 jI) \cdot \det\left[(R + j\omega L + \epsilon_0 jI) + P\left(j\omega C + \epsilon_0 jI\right)^{-1} P^T \right].$$

Consider the matrix $\left[(R + j\omega L + \epsilon_0 jI) + P\,(j\omega C + \epsilon_0 jI)^{-1}\,P^T\right]$

$$= R + j\omega L + j\epsilon_0 I \; - jP\,(\omega C + \epsilon_0 I)^{-1}\,P^T,$$
$$= R + j\left[\omega L + \epsilon_0 I - P\,(\omega C + \epsilon_0 I)^{-1}\,P^T\right]. \tag{5.7}$$

Since R is SPD and $\left[\omega L + \epsilon_0 I - P\,(\omega C + \epsilon_0 I)^{-1}\,P^T\right]$ is symmetric, it follows that $\det(A + jB - \epsilon I)$ is nonzero. Hence the reduced system, which corresponds to the matrix in (5.3), is stable.

Consequently, system (4.2) is stable for any value of s. The reduced system after performing a rank-1 correction has the same structure as the original system; hence, the rank-1 reduced system is also stable. □

6 Passivity of the Super Node Algorithm

Physically, a stable system is a system whose solutions will stay bounded with time for bounded input; this corresponds mathematically to a system whose eigenvalues are such that the solutions to the homogeneous equations contain damped exponential functions. The contribution of these solutions to the complete solution will then diminish on a characteristic time scale, the so-called transient time scale.

A *passive* system is a stable system which contains only components that absorb energy, such that the system encounters no power gain. Components may consume energy or store energy but may not produce energy or direct energy into the system from outside.

Passivity is a 'strong' property, both in the sense that passive systems are stable, but stable systems are not necessarily passive, but also because of the following:

Theorem 6.1 *Passivity is an additive property: If two passive systems are interconnected, the combined system will be passive and therefore stable.*

Note that a similar theorem does not hold for stability; if two separately stable systems are interconnected, the combined system will not necessarily be stable.

The mathematical criterion for passivity is that the appropriate transfer function, which can be derived from the Laplace transform which changed the system of differential equations describing the circuit into a system of algebraic equations (as described in Section 4), taken as a function of $j\omega$, has positive real part.

For the case of node reductions, one may ask: Will any reduction, i.e., one node being removed from the system, conserve passivity, and therefore stability?

In an extension of this report, the group is working to prove passivity of the super node algorithm.

7 Physical Realisability

In this section, we are concerned with the elimination of nodes occurring in the super node algorithm. We would like to prove that after such an elimination, the reduced system is realizable, i.e., satisfies Kirchhoff's laws. In the non resistive case, Kirchhoff's equations are

$$RI - PV - j\omega LI = 0, P^T I - j\omega KV = J,$$

where j denotes the square root of -1, ω is a positive constant, L is an $m \times m$ inductance matrix, K an $n \times n$ capacitance matrix, and all the other parameters and unknowns have the same meaning as before. The system of which V is the "solution" is now

$$\left(P^T(R - j\omega L)^{-1}P - j\omega K\right) V = J.$$

Assume that we wish to remove from the network some nodes that are not super nodes. If we denote this set of nodes by U_1, the set of nodes adjacent to them by U_2, and the set of remaining nodes (including all the super nodes) by U_3, we can decompose the network and the edge-node incidence matrix in exactly the same way as in the previous section. Indeed, U_2 is a cutset in the present case also.

We let K_i denote the capacitance submatrix corresponding to U_i for $i \in \{1, 2, 3\}$, and let T_i denote the matrix $R_i + j\omega L_i$ for $i \in \{1, 2, 3, 4, 5\}$. Then the system

$$\left(P^T(R - j\omega L)^{-1}P - j\omega K\right) V = J$$

can be rewritten as

$$M_{interconnect} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix},$$

where

$$M_{interconnect} = \begin{pmatrix} N & B^T T_2^{-1} C & 0 \\ C^T T_2^{-1} B & Q & E^T T_4^{-1} F \\ 0 & F^T T_4^{-1} E & S \end{pmatrix}.$$

Here $N = A^T T_1^{-1} A + B^T T_2^{-1} B - j\omega K_1$, $Q = C^T T_2^{-1} C + D^T T_3^{-1} D + E^T T_4^{-1} E - j\omega K_2$, and $S = F^T T_4^{-1} F + G^T T_5^{-1} G - j\omega K_3$.

Let us denote by A' the inverse of $A^T T_1^{-1} A + B^T T_2^{-1} B - j\omega K_1$. Then x is equal to $A'\left(r_1 - B^T T_2^{-1} C y\right)$, and the replacement of x in the middle part of the system yields

$$\begin{aligned}\left(-C^T T_2^{-1} B A' B^T T_2^{-1} C + C^T T_2^{-1} C + D^T T_3^{-1} D - j\omega K_2\right) y + E^T T_4^{-1} E y + E^T T_4^{-1} F z \\ = r_2 - \left(C^T T_2^{-1} B A'\right) r_1.\end{aligned}$$

Let H be the incidence matrix of any complete graph on the set U_2. One wishes to compute a matrix R_6 of resistances, a matrix L_6 of inductances, and a matrix K_4 of capacitances such that

$$-C^T T_2^{-1} B A' B^T T_2^{-1} C + C^T T_2^{-1} C + D^T T_3^{-1} D - j\omega K_2 = H^T \left(R_6 + j\omega L_6\right)^{-1} H - j\omega K_4.$$

Note that if we decide to remove the nodes one at a time, we may assume that A is vacuous (there is no edge in the subgraph induced by a single node, say, u). Also we may assume that all the edges incident to u are directed away from u. Then the matrix B is a vector all of whose entries equal 1, and C is the identity matrix. These assumptions, made without loss of generality, should simplify the calculations. In an extension of this report, the group is working to prove that the reduced system is physically realisable.

8 Conclusions

In conclusion, we were able to make progress along two directions. First, for the substrate, where we needed to drastically reduce a large resistive network, a graph-theoretic algorithm has been formulated. This algorithm includes two new features: (1) the elimination of large sets of internal nodes that are far from external nodes and (2) the use of cutsets. Our future work will focus on implementing our graph-theoretic algorithm and comparing its performance to existing techniques for model order reduction on NXP graph networks.

Second, for the interconnect structures, a thorough investigation has been performed to see how stability and passivity can be preserved whenever non-super nodes (and corresponding current branches) are eliminated. This is an entirely new way of looking at the problem. Previous techniques were based mainly on Krylov subspace and truncated balanced realization methods from numerical linear algebra and systems and control theory, respectively, whereas current research is focussed on linear problems with many inputs or specific structures, parameterized model order reduction, techniques for coupled problems, and methods for nonlinear problems [4, 5]. We proved that the resulting reduced systems are all stable and passive. Future work needs to concentrate on investigating the nonzero off-diagonal blocks in the $\begin{pmatrix} L & 0 \\ 0 & C \end{pmatrix}$ matrix. It may happen that these blocks end up being 0 automatically if one makes the correct choice of nodes and/or branches to be deleted from the network.

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