ATMOSPHERIC SOUND PROPAGATION

1. Introduction

The National Acoustic Laboratory (NAL), is interested in developing a reliable model of sound propagation in the atmosphere. Such a model would be useful to them for determining whether noise producers should go ahead with their activities at a particular time, or wait until more favourable meteorological conditions prevail, so as to minimise annoyance to neighbours. The propagation of sound in the atmosphere is affected by variations in air density, causing variations in the local sound speed, as well as by air motion, *i.e.* wind, which is likely to be turbulent. In this instance, NAL is interested in sound propagation over distances of kilometres, in which case the effects of the ground covering on sound propagation are not important. NAL has the capacity to measure air temperature and humidity, in order to determine the local sound speed, and also can obtain some information about wind velocity.

It was pointed out to the mathematicians that occasionally the meteorological conditions are such that the noise from a particular event may be much louder than expected. The increased sound level is often attributed to the prevailing wind, but may also be caused by temperature effects. The NAL representative, Mr Warwick Williams, indicated that such an anomaly often occurs when there is a temperature inversion, that is, when the air temperature increases with height, which is opposite to the normal situation. The existence of an anomalous increase in local sound intensity was noted to be reminiscent of the formation of a "caustic", caused by interference between groups of sound waves. It is noteworthy that the existence of caustics for the propagation of sound in the earth has been studied by geophysicists, and the identification of caustics is useful because they carry a large amount of energy (and hence sound).

Thus the major activity undertaken during the three working days of the study group was to develop a method of determining the existence and position of caustics for given atmospheric conditions. This helped to satisfy the requirements of NAL, because the relevant model equations were determined and discussed, and a useful measure of sound intensity was obtained. Additionally, the equations developed have allowed us to isolate specific conditions under which caustic phenomena will occur.

2. Geometrical acoustics

There is a large collection of literature dealing with the subject of geometrical acoustics. Two particularly useful textbooks are by Landau & Lifshitz (1959) and

Lighthill (1978). Both give equations for the tracing of sound "rays". They are based on the assumption that the amplitude and direction of propagation of a sound wave vary only slightly over distances of the order of its wavelength. Then a sound wave can be regarded as a plane wave in any small region of space.

We consider a sound wave which takes the form

$$A\exp[i(\mathbf{k}.\mathbf{r}-\omega t)]. \tag{1}$$

The wave vector, $\mathbf{k}$, and the frequency ω are connected by a dispersion relation. In steady propagation of sound in an inhomogeneous medium, at rest, the dispersion relation is

$$\omega = |\mathbf{k}|c(x,y,z) \tag{2}$$

where $c(x,y,z)$ is the local sound speed in the medium (the atmosphere, in this case). In a medium moving with velocity $\mathbf{u}$, the dispersion relation is

$$\omega = |\mathbf{k}|c + \mathbf{u}.\mathbf{k} \tag{3}$$

The second term in equation (3) is a "Doppler effect" term, which accounts for a frequency shift due to the moving medium. The ray equations can be determined from the dispersion relations by using

$$\frac{d\mathbf{r}}{dt} = \frac{\partial\omega}{\partial\mathbf{k}}, \tag{4}$$

$$\frac{d\mathbf{k}}{dt} = -\frac{\partial\omega}{\partial\mathbf{r}}. \tag{5}$$

Equations (4) and (5), along with a dispersion relation, are the basic equations of geometrical acoustics. By integrating with respect to time, t, or equivalently distance along individual rays, one can trace rays in any given atmospheric condition, and thus obtain a qualitative picture of the sound propagation. It must be remembered that the results represent the solution for a single frequency sound wave in a non-turbulent atmosphere.

A measure of sound intensity is given by the determinant of the Jacobian

$$J(\boldsymbol{\alpha},t) = \left|\frac{\partial(\mathbf{r})}{\partial(\boldsymbol{\alpha},t)}\right| \tag{6}$$

where $\boldsymbol{\alpha}$ is a set of parameters defining the initial direction of a ray. Because J is a measure of the local divergence of rays, it is inversely proportional to the sound intensity.

When rays get very close together the assumptions of geometrical acoustics break down. In such a situation, the sound waves interfere and lead to the formation of caustics. The quantity J therefore gives a means of locating the position of caustics, because $J=0$ corresponds to a caustic surface.

3. Atmospheric conditions

The atmosphere is typically in motion, and the air motion will be turbulent. Turbulence acts to attenuate sound, but the mechanisms of such attenuation were beyond the scope of those gathered for this session, due to time restrictions, so turbulence was not considered. A literature search carried out prior to the MISG found that some experimental and modelling work has been done on this problem (Ingard (1953), Daigle *et al.* (1978), Wenzel (1971)).

The sound speed in air is typically 340 ms^{-1} and a moderate air speed is 20 knots $\simeq$ 40 $\mathrm{km\,hr}^{-1} \simeq$ 17 ms^{-1}. Thus we see that in the atmosphere, typically

$$\frac{|\mathbf{u}|}{c} \ll 1. \tag{7}$$

To first order, this implies that a sound will reach its destination before the wind has a significant effect. Hence, as a first approximation, we neglect the air motion in the following considerations. This approximation is not too unreasonable, because the analysis is still appropriate to a moving medium, with the benefit that the calculations are much simpler.

Another important aspect of the atmosphere is that it is vertically stratified. That is, there are strong variations of the properties of the atmosphere in the vertical direction. On the scale of kilometres, the horizontal variation of meteorological properties is typically negligible. So we make the further simplification that the sound speed (and the air velocity) is only dependent on the vertical co-ordinate, z.

By assuming vertical stratification and *motionless air*, the ray equations are considerably simplified. In a Cartesian co-ordinate system they become

$$\frac{d\mathbf{r}}{dt} = \frac{d(x,y,z)}{dt} = \frac{\mathbf{k}}{|\mathbf{k}|}c(z), \tag{8}$$

$$\frac{d\mathbf{k}}{dt} = \frac{d(k,l,m)}{dt} = -|\mathbf{k}|(0,0,\frac{dc}{dz}). \tag{9}$$

This system is in fact two-dimensional, because the horizontal components of the wavevector are constant (from equation (9)) and because the remaining equations are dependent only upon the vertical co-ordinate. Thus we will neglect the equations for y and l in what follows. The problem reduces to the following system

$$\frac{dx}{dt} = \frac{k}{|\mathbf{k}|}c(z), \tag{10}$$

$$\frac{dz}{dt} = \frac{m}{|\mathbf{k}|}c(z), \tag{11}$$

$$\frac{dm}{dt} = -|\mathbf{k}|\frac{dc}{dz}, \tag{12}$$

along with the dispersion relation

$$\omega = |\mathbf{k}|c(z). \tag{13}$$

Here, k is a constant. Also, if the z-dependence of c is known, then the wavevector component m can be found from equation (13) to be

$$m = \pm\sqrt{\frac{\omega^2}{c^2} - k^2} \tag{14}$$

which is purely a function of z. Note that if k is specified for a ray, then the initial value of m is also specified by equation (13). The $\pm$ sign in (14) determines whether a ray is going upwards (+) or downwards (–). For a noise that occurs at ground level, all rays will initially be going upwards, and we are only interested in rays that turn over and return to ground level, since they are the ones that will cause annoyance. Those rays that turn over will have the property

$$\frac{\omega^2}{c^2} - k^2 = 0 \tag{15}$$

at some z value.

Because the right-hand-sides of equations (10), (11) are functions of z only, it is convenient to take z as the new independent variable. This gives

$$\frac{dx}{dz} = \frac{k}{m}, \tag{16}$$

$$\frac{dt}{dz} = \frac{|\mathbf{k}|}{mc}. \tag{17}$$

Equations (16) and (17) can be integrated to get $x(z)$ and $t(z)$. We also need the two-dimensional Jacobian

$$\begin{aligned} J &= \left|\frac{\partial(x,z)}{\partial(k,t)}\right| = \frac{dz}{dt}\left|\frac{\partial(x,t)}{\partial(k,z)}\right| \\ &= \frac{kc}{|\mathbf{k}|}\frac{\partial t}{\partial k} - \frac{\partial x}{\partial k} \end{aligned}$$

using equations (10-17). For certain simple functional dependences of c upon z it is possible to integrate (16), (17) explicity, and thus obtain an expression for the Jacobian. In such cases, the location of caustics can be determined explicitly. We carry out this process in the following two informative examples.

4. Sound speed varying linearly with height

Here, we model the sound speed by

$$c(z) = c_0 + c_1 z. \tag{18}$$

In this case, solution of (16) shows that rays are arcs of circles, so they take the form

$$(x - x_0)^2 + (z + \frac{c_0}{c_1})^2 = \frac{\omega^2}{c_1^2 k^2} \tag{19}$$

with

$$x_0 = \frac{1}{c_1}\sqrt{\frac{\omega^2}{k^2} - c_0^2}.$$

It is also possible to solve for $t(z)$ and then determine the Jacobian, which is found to be

$$J = \frac{\omega}{k^2 c_1} \frac{\cos\phi(\cos\phi_0 - \cos\phi)}{\cos\phi_0} \tag{20}$$

where $\phi(z)$ is defined by

$$\sin\phi(z) = \frac{k}{\omega}c(z) = \frac{k}{\omega}(c_0 + c_1 z) \tag{21}$$

and

$$\phi_0 \ = \ \phi(z=0) \ = \ \sin^{-1}(\frac{kc_0}{\omega}). \tag{22}$$

It is simple to draw a picture of the rays generated by (19), since they are simply arcs of circles, and such a picture is shown in Figure 1. In this example, we have set $\omega = c_0 = c_1 = 1$ for simplicity. It is clear that the rays never overlap, except at the origin, so that the origin (ie the sound source) is the only possible location of a caustic. This corresponds to setting $\phi = \phi_0$ in equation (20), so that $J = 0$. There is also a zero of J when $\cos\phi = 0$, which happens when a ray becomes horizontal. At such a point, equation (17) is singular and the analysis breaks down. Consequently, this point does not correspond to the location of a caustic.

5. An inversion in the atmosphere

Here, suppose the sound speed is

$$c(z) = \begin{cases} c_0 + c_1 z & z \le z_1 \\ c_0 + c_1 z_1 + c_2(z - z_1) & z > z_1 \end{cases} \tag{23}$$

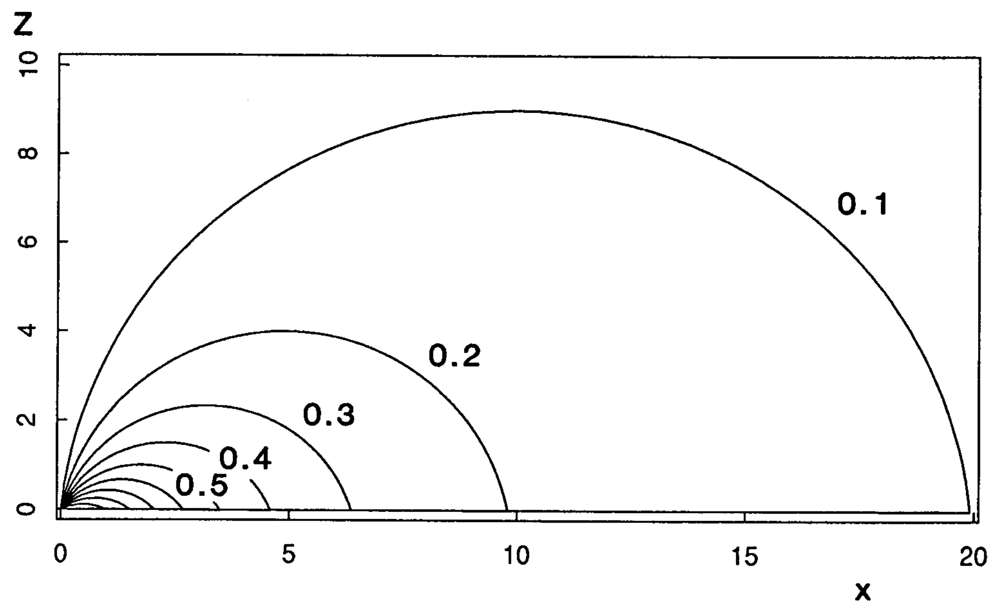

Figure 1: Numerically determined sound rays for a linearly varying sound speed. The values marked on each curve are the horizontal component of the wavenumber k.

with $c_1 < 0$ and $c_2 > 0$, thereby describing an "inversion". A ray in this case will consist of arcs of circles, as shown in Figure 2. Thus, for each piece of arc, the expressions for Section 4 are valid, and they allow determination of the Jacobian.

For the part of the arc labelled (3), the expressions for $x(z,k), t(z,k)$ and $J(z,k)$ are

$$x(z,k) = \frac{\omega}{kc_0}\{\cos\phi_0 + 2(\frac{c_1}{c_2} - 1)\cos\phi_A + \cos\phi(z)\}, \tag{24}$$

$$t(z,k) = \frac{1}{c_1}\{2(1 - \frac{c_1}{c_2})\ell n|\tan\frac{\phi_A}{2}| - \ell n|\tan\frac{\phi_0(z)}{2}| - \ell n|\tan\frac{\phi(z)}{2}|\}, \tag{25}$$

$$J(z,k) = \frac{\omega}{k^2 c_1}\cos\phi(z)\{1 + \frac{\cos\phi}{\cos\phi_0} + 2(\frac{c_1}{c_2} - 1)\frac{\cos\phi}{\cos\phi_A}\}, \tag{26}$$

where

$$\sin\phi(z) = \frac{k}{\omega}(c_0 + c_1 z) \tag{27}$$

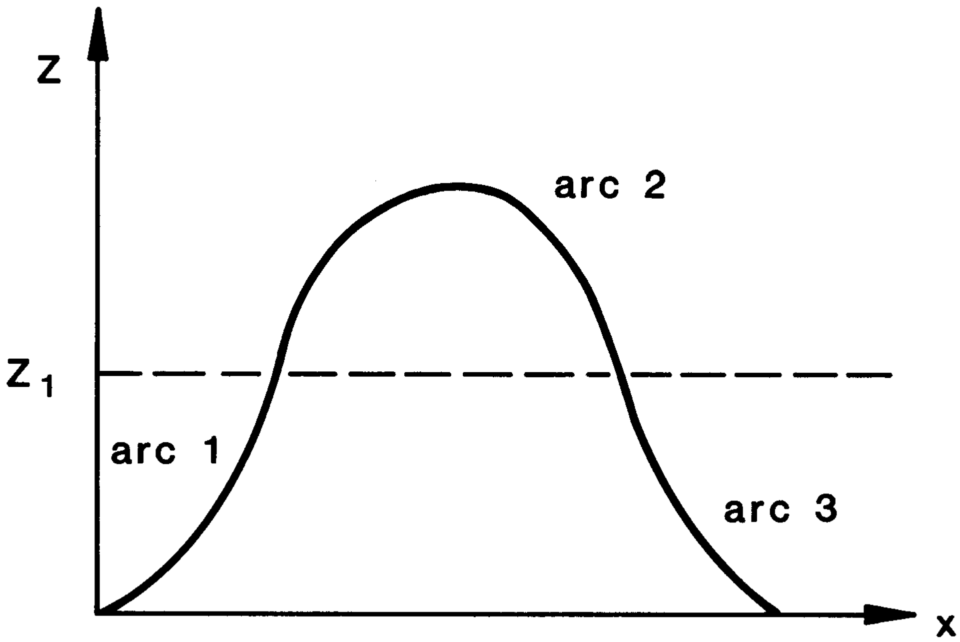

Figure 2: Schematic of a sound ray in the case of a simple inversion. The ray consists of three separate circular arcs.

and hence

$$\cos\phi_0 = \sqrt{1 - \frac{k^2 c_0^2}{\omega^2}} \;\; ; \;\; \cos\phi_A = \sqrt{1 - \frac{k^2}{\omega^2}(c_0 + c_1 z_1)^2} \tag{28}$$

where z_1 is given by equation (23). From equation (26) we see that $J = 0$ when

$$1 + \frac{\cos\phi}{\cos\phi_0} + 2\left(\frac{c_1}{c_2} - 1\right)\frac{\cos\phi}{\cos\phi_A} = 0$$

or

$$\cos\phi = \sqrt{1 - \frac{k^2}{\omega^2}c^2(z)} = \frac{\cos\phi_0 \cos\phi_A}{2(1 - c_1/c_2)\cos\phi_0 - \cos\phi_A} \tag{29}$$

which defines the height z at which a ray with parameter k touches the caustic surface. In particular, we are interested in where the caustic hits the ground, since this would be the cause of a loud noise. Thus we substitute $z = 0$ into (29), and rearrange to find the particular ray which touches the caustic at ground level. This gives

$$k^2 = \omega^2 \frac{(2c_2 - c_1)}{c_2^2 z_1(c_1 z_1 + 2c_0) + c_0^2(2c_2 - c_1)} \tag{30}$$

which can then be substituted into (24) to give the x value at which the caustic strikes the ground.

In order to verify this result, the original equations (10-12) have been integrated numerically, using the IMSL routine DGEAR, for a range of k values. For simplicity (and in the absence of knowledge of real data) the following values of parameters were used

$$\begin{aligned} \omega &= 1, \\ c_0 &= 1, \\ c_1 &= -1, \\ c_2 &= 4, \\ z_1 &= 0.5. \end{aligned}$$

The value of k given by (30) is $\sqrt{3/7} \simeq 0.655$ and (24) gives

$$x(0, 0.655) \simeq 1.30$$

From the graph of Figure 3, which is the numerical solution, we see that there is a caustic surface hitting the ground at $x \simeq 1.3$ and close inspection shows that the particular ray touching the caustic occurs for k between 0.6 and 0.7.

6. Numerical solution

The numerical solution of the ray equations should be quite straightforward, and indeed Figure 3 has resulted from simple application of an IMSL routine. It is also possible to produce a set of o.d.e.'s for the components of the Jacobian, J. Then in three dimensions, it is necessary to solve 10 first order o.d.e.'s in order to find the sound intensity. Of course, if one neglects the effects of wind, then this reduces to only 5 o.d.e.'s. The necessary software to do this on a PC type machine may be available in the public domain, and is definitely available commercially (*e.g.* from NAG). By solving these equations, it would be possible to produce a map of relative sound intensity at ground level from a single frequency source. The results obtained will only be qualitative, however, because of effects such as turbulence and because most sources produce a band of frequencies. Only comparison with field measurements can show how useful such a model would be.

7. Conclusions

The geometrical acoustics approach described here has enabled the development of a numerical procedure for tracing sound rays and determining local sound

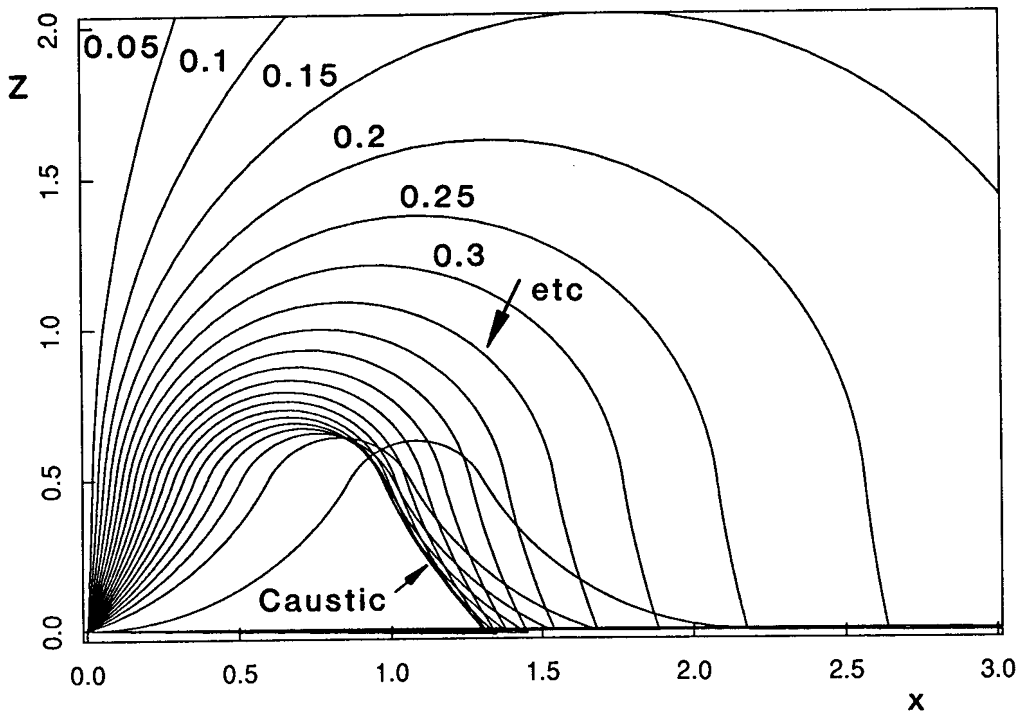

Figure 3: Numerically determined sound rays for a simple inversion. The values marked on each curve are the horizontal component of the wavenumber k.

intensity. The actual mechanics of the numerical solution have not been considered, since it is envisaged that there is no significant mathematical difficulty involved. Instead, we have examined the relevant equations and developed a method for determining the existence and position of caustics. The idea that unusually loud sounds can be heard when an inversion exists is verified by the second case study performed, because we showed that a caustic surface can exist in such a case. Also, we have questioned the importance of wind upon the propagation of sound in the atmosphere, due to the relative magnitudes of typical air speeds and sound speed. We have also pointed out that the results from a ray tracing model will only be qualitative, due to the effects of turbulence, the spread of frequencies produced by most sources and the effects of interference.

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