Mixing in the downward displacement of a turbulent wash by a laminar spacer or cement slurry

Schlumberger Dowell

6th to 9th April 1999

1 Introduction

When drilling an oil well, the well is lined by sinking a steel casing, or liner, into the drilling mud (see Figure 1). The inside of the liner forms a central pipe, and leaves an annular gap between the outside of the liner and the surrounding rock. The inside and outside of the liner are initially filled with drilling mud, which is displaced by pumping a sequence of fluids down inside the liner from the surface. Typically, a chemical wash is pumped down first, followed by a spacer, and finally a cement slurry. The wash, which is usually water based, is less dense than the spacer, a water-based suspension, which is itself less dense than the cement slurry.

The scenario of a light wash being forced down a pipe by a denser spacer fluid is liable to instabilities of Rayleigh–Taylor type. The two fluids are miscible, so mixing between the wash and the spacer is likely to take place. This may impair the intended efficiency of the wash in displacing mud from the walls of the annulus.

Schlumberger Dowell asked the study group to estimate the amount of mixing between the wash and the spacer, and its dependence on parameters such as the pumping rate, diameter of the pipe, viscosity of the fluids etc. In particular, they asked us to consider the following questions:

- Is there any physical mechanism that might prevent mixing between the wash and spacer/slurry?
- Can we give a reliable estimate of the length of mixed region?
- How does the above vary with pump rate, tube diameter, etc?
- What happens if we stop pumping during downward displacement?

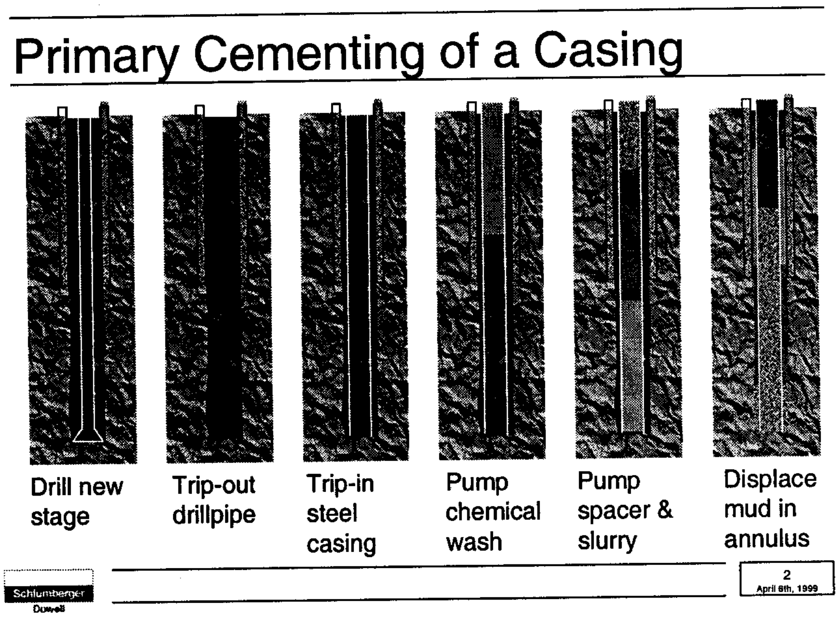
2 Flow regimes

Concentrating on the downward flow inside the pipe, the study group identified four different flow regimes of interest.

1. Initial mixing across the pipe.
2. Spanwise mixing completed, what streamwise stratification can be stabilised against Rayleigh–Taylor instabilities?
3. Widening of the mixed layer by streamwise dispersion.
4. Effects of halting pumping.

The study group tried to estimate typical lengthscales and timescales associated with each regime in turn.

Figure 1: Stages in drilling an oil well. The study group considered the fifth stage, in which the wash is being displaced by denser fluids pumped from above.



2.1 Typical parameters

To fix ideas we considered some typical parameters, namely a pipe radius $a = 0.1$ m, a pumping velocity of $\bar{u} = 1\ \mathrm{m\,s^{-1}}$, a spacer viscosity ten times that of water ($\nu = 10^{-5}\ \mathrm{m^2\,s^{-1}}$), and a density ratio $\Delta\rho/\rho = 0.3$. We also assumed that the pipe would be nearly vertical, since shall find that most of the mixing takes place near the top of the pipe. See figure 2(a). A wider range of typical parameters are given below.

Typical material properties:

	density $\rho/\mathrm{kg\,m^{-3}}$	viscosity $\nu/\mathrm{m^2\,s^{-1}}$
wash	900 to 1100	9×10^{-7} to 5.5×10^{-6}
spacer	1200 to 1600	1.8×10^{-5} to 4.1×10^{-4}

Typical pipe radii and pumping rates:

radius a/m	mean velocity $\bar{u}/\mathrm{m\,s^{-1}}$	Re ($\nu = 10^{-5}\ \mathrm{m^2\,s^{-1}}$)
0.05	0.63 to 3.2	2000 to 16000
0.11	0.52 to 0.79	5700 to 8700
0.16	0.24 to 0.5	3800 to 8000

The Reynolds number in the wash may be expected to be somewhat higher due to its lower viscosity.

Pipes are between 500 m and 5000 m in length. A typical volume of spacer corresponds to between 100 m and 2000 m of pipe.

2.2 A simple model for turbulent pipe flow

The Reynolds number based on the pumping velocity, $Re = a\bar{u}/\nu$, is high enough for the flow to be turbulent; $Re = 10^4$ for the above typical parameters ($\bar{u} = 1\ \mathrm{m\,s^{-1}}$, $a = 0.1$ m, $\nu = 10^{-5}\ \mathrm{m^2\,s^{-1}}$). A 'universal' model for turbulent flow of a homogeneous Newtonian fluid in a straight cylindrical pipe may be found in §154 of Goldstein (1938) or §5.6 of Townsend (1976). A key parameter is the friction velocity $u^\star = \sqrt{\sigma_{\text{wall}}/\rho}$, where σ_{wall} is the stress on the pipe wall, and ρ is the density. The mean streamwise velocity $\bar{u}$ is related to the friction velocity $u^\star$ via the equation

$$\frac{\bar{u}}{u^\star} = \frac{1}{k}\log Re^\star + 5.5, \text{ where } Re^\star = \frac{au^\star}{\nu} = Re\frac{u^\star}{\bar{u}}. \tag{1}$$

$Re^\star$ is the Reynolds number based on the friction velocity $u^\star$, $k = 0.4$ is von Karman's constant, and 5.5 is another empirical constant. Equation (1) is often rewritten in terms of the Fanning friction factor $fr = 2u^{\star 2}/\bar{u}^2$,

$$fr^{-1/2} = 4\log_{10}(Re\ fr^{1/2}) - 0.4, \tag{2}$$

as in Table A3 of Nelson (1990) or equation (19) of Goldstein (1938, §154), though the latter uses γ instead of fr. For $Re \lesssim 10^5$, Blasius proposed an approximate explicit formula for fr, (equation 20 in Goldstein (1938) §155)

$$fr = 0.0665(\bar{u}a/\nu)^{-1/4} = 0.0665 Re^{-1/4}. \tag{3}$$

The associated eddy viscosity D is given by

$$D = ku^\star(a - r), \tag{4}$$

close to the pipe walls, where r is the radial coordinate (*e.g.* Townsend (1976) §5.6). The eddy viscosity is expected to be uniform in the bulk of the flow, where we used the value $D = ku^\star a$.

Although the study group only considered Newtonian fluids, analogous equations exist for power-law and Bingham fluids (*e.g.* tables A-4 and A-5 of Nelson (1990)).

3 Initial mixing across the pipe

We supposed that the denser spacer fluid would tend to form a finger running down the side of the well, driving an upward return flow of the less dense wash, as shown in figure 2(b). This scenario resembles a turbulent gravity current running down the lower side of the pipe. We assumed that this flow would reach a terminal finger velocity u_f in which buoyancy is balanced by the turbulent stress σ_{wall} on the wall. We shall see below that this terminal velocity is an order of magnitude larger than the pumping velocity, so we ignored the pumping velocity in this calculation. Taking $g = 10\ \mathrm{m\,s^{-2}}$, $\Delta\rho = 300\ \mathrm{kg\,m^{-3}}$ and $a = 0.1$ m, the wall stress is

$$\sigma_{\text{wall}} = ga\Delta\rho/2 \sim 150 \text{ Pa}, \tag{5}$$

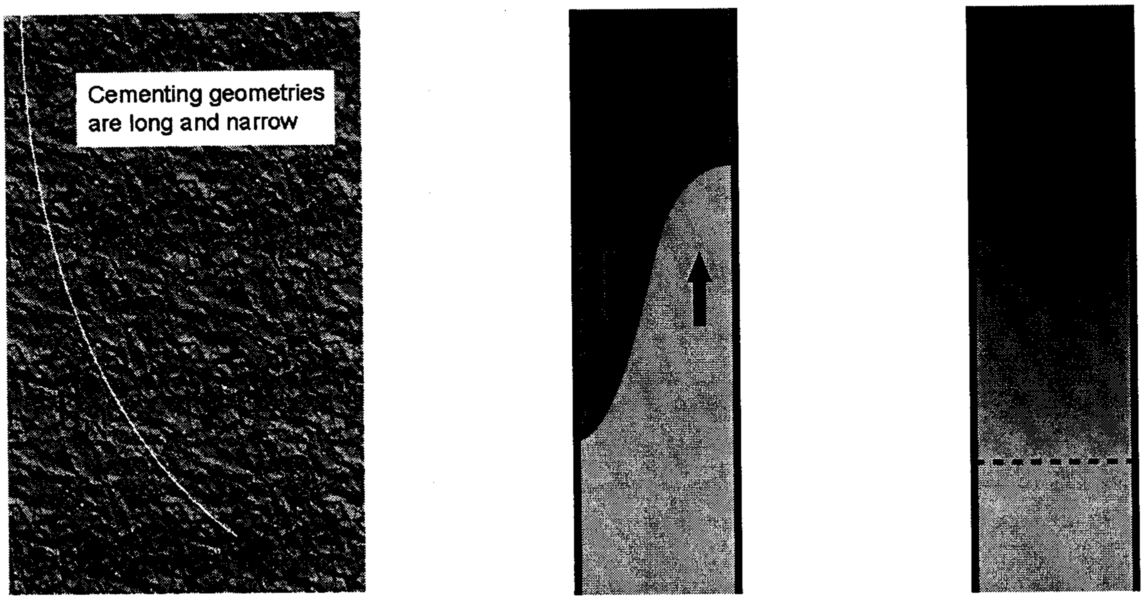
and the friction velocity is

$$u^\star = \sqrt{\sigma_{\text{wall}}/\rho} = \sqrt{ga\Delta\rho/2\rho} \sim \sqrt{150\ \text{Pa}/1000\ \text{kg}\,\text{m}^{-3}} \sim 0.4\ \mathrm{m\,s^{-1}}. \tag{6}$$

From equation (1), the friction Reynolds number $Re^\star = au^\star/\nu \sim 4000$, and the finger velocity u_f (equated with $\bar{u}$) is around $10\ \mathrm{m\,s^{-1}}$. This is an order of magnitude larger than the pumping velocity, which is itself large enough for the flow to be turbulent. The corresponding eddy viscosity is

$$D = kau^\star = 0.2\sqrt{2ga^3\Delta\rho/\rho} \sim 10^{-2}\ \mathrm{m^2\,s^{-1}}, \tag{7}$$

Figure 2: (a) A typical well. We assume most of the mixing takes place near the top where the pipe is nearly vertical. (b) Initial mixing via a turbulent gravity current. (c) Streamwise mixing of a horizontally uniform mixture.



(a) (b) (c)

which is about 1000 times the molecular viscosity ν. We assumed that D is also the eddy diffusivity for mixing between the spacer and the wash. The diffusion time T for turbulent mixing across the pipe is thus

$$T = a^2/(2D) = 1.25\sqrt{2a\rho/(g\Delta\rho)} \sim 1 \text{ s}, \tag{8}$$

which is independent of the molecular viscosities of the two fluids. In fact, this estimate holds even for the worst case scenario ($a = 0.16$ m, $\Delta\rho = 100 \text{ kg m}^{-3}$). We conclude that turbulent mixing will erase any variations in density across the pipe within a few seconds, by which time the flow will have descended at most a few tens of metres from the inlet.

4 Stabilising streamwise stratification

As the suspending fluid in the spacer is miscible with water, we modelled the spacer/wash mixture as a single fluid with a spatially varying concentration C of heavy particles. Thus pure spacer fluid would correspond to $C = 1$, say, and pure wash to $C = 0$. As argued above, we expect turbulent mixing to establish a uniform concentration across the pipe within a few tens of metres from the inlet. We thus considered a mean concentration profile $C(z)$ depending only on the streamwise coordinate z, as shown in figure 2(c).

This concentration profile appears at rest in a frame moving with the pumping velocity $\bar{u} \sim 1 \text{ m s}^{-1}$. The governing equations in this moving frame, in the Boussinesq approximation, are

$$\frac{\partial C}{\partial t} + \mathbf{u}\cdot\nabla C = \nabla\cdot(\kappa\nabla C), \tag{9a}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla(p/\rho) + \alpha C\mathbf{g} + \nabla\cdot(\nu\nabla\mathbf{u}), \quad \nabla\cdot\mathbf{u} = 0. \tag{9b}$$

Here $\mathbf{u}$ is the fluid velocity, $\alpha = (1/\rho)\mathrm{d}\rho/\mathrm{d}C$ is the expansion coefficient, and κ and ν are the possibly space-dependent concentration diffusivity and kinematic viscosity respectively. In the Boussinesq approximation we neglect variations in density, and other material properties, except for the variation in density in the buoyancy term $\alpha C\mathbf{g}$ in the momentum equation. See Chandrasekhar (1961) §8 or Drazin & Reid (1981) §7.2.

As the flow is turbulent, according to §2.2, we followed Reynold's analogy and used the same turbulent value D for both κ and ν. Equivalently, we assumed the Prandtl number to be one. Having replaced κ and ν by a turbulent effective value D, we assumed that (9a,b) describe the mean part of the turbulent flow in some locally averaged sense. Making the usual decomposition $\mathbf{u} = \langle\mathbf{u}\rangle + \mathbf{u}'$ and $C = \langle C\rangle + C'$, we identified $\mathbf{u}$ and C in (9a,b) with their local averages $\langle\mathbf{u}\rangle$ and $\langle C\rangle$, and assumed that the small scale fluctuations $\mathbf{u}'$ and C' only influence the mean flow by contributing to the turbulent eddy diffusivity D.

In the Boussinesq approximation we also assume that the molecular diffusivity, needed in (1), is spatially uniform. As the turbulent diffusivity D depends only appears logarithmically on the molecular viscosity ν via the definition of $Re^\star$ in (1), or as $D \propto \nu^{-1/8}$ from Blasius' formula (3), this approximation should be reasonable even though ν itself will vary according to the local concentration of particles from the spacer.

In the frame moving with the pumping velocity $\bar{u}$, the noslip boundary condition implies that the mean fluid velocity should be $-\bar{u}\hat{\mathbf{z}}$ on the pipe wall. Thus we expect a mean profile of the form $\mathbf{u} = U(r)\hat{\mathbf{z}}$ with $U \approx 0$ over most of the pipe, but with a shear layer near the wall so that $U = -\bar{u}$ at the wall. However, we ignored the shear layer and assumed plug flow, $\mathbf{u} = 0$ in the moving frame, so we could formulate a tractable stability problem. As the eddy diffusivity $D = ku^\star(a-r)$ vanishes at the pipe wall $r = a$, this provides some justification for ignoring the shear layer on the wall, and adopting what are effectively free slip boundary conditions.

Thus the pumping velocity enters the problem only in supplying the mean velocity $\bar{u}$ for the turbulence model, from which we determine $u^\star$ and D by solving equation (1).

4.1 Uniform streamwise stratification

By analogy with the Rayleigh-Bénard problem for thermal convection (*e.g.* Chandrasekhar (1961)) we considered the linear stability of a uniformly stratified layer, $C = C_0 + z/L$, where L is the scale height. The key dimensionless parameter is the Rayleigh number,

$$Ra = g\frac{\Delta\rho}{\rho L}\frac{a^4}{D^2} = g\frac{\Delta\rho}{\rho L}\frac{a^2}{k^2 u^{\star 2}}, \tag{10}$$

where a is the pipe radius, and D the turbulent diffusivity. The stratified layer is expected to be stable provided the Rayleigh number is below some critical value $Ra < Ra_{\text{crit}}$. We expect that the lengthscale L for the mixed layer will be that for which the Rayleigh number defined in (10) attains this critical value,

$$L = \frac{1}{Ra_{\text{crit}}}\frac{g\Delta\rho}{\rho}\frac{a^4}{D^2} = \frac{1}{Ra_{\text{crit}}}\frac{g\Delta\rho}{\rho}\frac{a^2}{k^2 u^{\star 2}}. \tag{11}$$

This idea seems to have been proposed first by Taylor (1954b), and is supported by experiments reported by Lowell & Anderson (1982) and Taylor (1954b).

Letting $C = C_0 + z/L + C'$ and $\mathbf{u} = \mathbf{u}'$, and neglecting quadratic terms in $\mathbf{u}'$ and C', we obtain

$$\frac{\partial C'}{\partial t} + w'/L = \nabla\cdot(D\nabla C'), \tag{12a}$$

$$\frac{\partial \mathbf{u}'}{\partial t} = -\nabla(p'/\rho) + \alpha\mathbf{g}C' + \nabla\cdot(D\nabla\mathbf{u}'), \quad \nabla\cdot\mathbf{u}' = 0. \tag{12b}$$

In dimensionless variables, and with a spatially uniform D,

$$\frac{\partial C'}{\partial t} + w' = \nabla^2 C', \text{ and } \frac{\partial \mathbf{u}'}{\partial t} = -\nabla p' - Ra\,\hat{\mathbf{z}}\,C' + \nabla^2\mathbf{u}', \tag{13}$$

where the Rayleigh number is defined in (10). The pressure and the horizontal velocity components may be eliminated by taking $\nabla\times\nabla\times$ of the perturbation momentum equation,

$$\frac{\partial}{\partial t}\nabla^2\mathbf{u}' = Ra\nabla\times\nabla\times(\hat{\mathbf{z}}\,C') + \nabla^4\mathbf{u}', \tag{14}$$

and then taking the vertical component [Drazin & Reid (1981) §8.1, Chandrasekhar (1961) §9],

$$\frac{\partial}{\partial t}\nabla^2 w' = -Ra\nabla_\perp^2 C' + \nabla^4 w', \tag{15}$$

where $\nabla_\perp^2 = \partial_{xx} + \partial_{yy} = \nabla^2 - \partial_{zz}$ is the horizontal Laplacian. In other words, vertical density gradients may be balanced by pressure gradients, so only horizontal density gradients remain to drive an instability. Equations (13) and (15) now form a closed system for C' and w', equivalent to equations (8.8,8.13) of Drazin & Reid (1981), or (74,76) of Chandrasekhar (1961) §9, on replacing C' by $-\theta'$.

4.2 An eigenvalue problem for the Rayleigh number

We assumed that these equations become unstable via an exchange of stabilities, as in Rayleigh-Bénard convection between parallel planes [Drazin & Reid (1981) §9.1, Chandrasekhar (1961) §11]. In other words, the time derivatives vanish at the critical Rayleigh number Ra_{crit}.

The onset of Rayleigh-Bénard convection in a cylindrical pipe has been studies by Yih (1959) and Batchelor & Nitsche (1993). The most unstable mode is independent of z [Proctor 1993] and proportional to $\cos\theta$ in azimuth [Yih 1959]. A more complex mode, such as one with z dependence, would encounter more dissipation, and would only become unstable for Rayleigh numbers larger than the critical Rayleigh number for the mode considered here. Perhaps surprisingly, the same is true for axisymmetric modes [Yih 1959].

The assumption of lack of z dependence, and also the boundary conditions for the eigenvalue problem, only hold for a vertical pipe, so that $\hat{\mathbf{z}}$ and $\hat{\mathbf{g}}$ are antiparallel. We assumed that a stable mixed layer will be established close to the top of the pipe, after the initial horizontal mixing of §3, where the pipe is still very close to vertical (see figure 2(a)).

With these assumptions, (13) and (15) reduce to an eigenvalue problem for the perturbations w' and C' in the streamwise velocity and concentration respectively,

$$\mathcal{L}C' = w', \text{ and } \mathcal{L}w' = Ra\,C', \text{ where } \mathcal{L}f = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(rD(r)\frac{\mathrm{d}f}{\mathrm{d}r}\right) - \frac{f}{r^2}. \tag{16}$$

The lack of z-dependence makes $\nabla_\perp^2$ identical to ∇^2, so we may 'cancel' one Laplacian in (15).

The boundary conditions are w', C' bounded as $r \to 0$, $\mathrm{d}C'/\mathrm{d}r = 0$ at $r = 1$ (no flux of particles) and either $w' = 0$ at $r = 1$ (rigid) or $\mathrm{d}w'/\mathrm{d}r = 0$ at $r = 1$ (free slip). This eigenvalue problem has an analytical solution for rigid boundaries [Yih 1959],

$$w' = [J_1(kr)I_1(k) - I_1(kr)J_1(k)]\cos\theta, \text{ with } Ra_{\text{crit}} = k^4 = 67.94. \tag{17}$$

This critical Rayleigh number first appeared in Taylor (1954b). The corresponding solution for free slip boundaries is

$$w' = [J_1(kr)I_1'(k) - I_1(kr)J_1'(k)]\cos\theta, \text{ with } Ra_{\text{crit}} = k^4 = 11.49. \tag{18}$$

We also obtained the upper bounds $Ra_{\text{crit}} \leq 71.68$ and $Ra_{\text{crit}} \leq 11.51$ respectively by considering the trial functions $C' = r - r^3/3$, and $w' = r - r^3$ or $w' = r - r^3/3$, in a variational formulation of (16) (see §4.4).

Although we allow for a spatially dependent eddy diffusivity $D(r)$ in (16), the study group only considered uniform diffusivities, for which the eigenproblem is analytically tractable. The free slip boundary conditions were an attempt to compensate for the eddy diffusivity vanishing at the walls according to (4).

4.3 Stability estimates

We used (10) to obtain an estimate for the minimum length L, above which a mixed layer should be stable against Rayleigh–Bénard convection. Using Blasius' approximate formula (3) to obtain an explicit expression for $u^\star$, we find that

$$L = \frac{1}{Ra_{\rm crit}} g \frac{\Delta\rho}{\rho} \frac{a^2}{k^2 u^{\star 2}} \approx \frac{1}{Ra_{\rm crit}} g \frac{\Delta\rho}{\rho} \frac{a^2}{k^2 \bar{u}^2} \frac{2}{0.0665} \left(\frac{\bar{u}a}{\nu}\right)^{1/4} \quad \text{for } Re = \frac{\bar{u}a}{\nu} \lesssim 10^5. \tag{19}$$

Using the free-slip critical Rayleigh number, $Ra_{\rm crit} = 12$, and a vertical pipe ($g = 10\ \mathrm{m\,s^{-2}}$), this becomes

$$L = 160 \frac{\Delta\rho}{\rho} \frac{a^2}{\bar{u}^2} \left(\frac{\bar{u}a}{\nu}\right)^{1/4} \text{ metres} \approx 1600\, \frac{\Delta\rho}{\rho} \frac{a^2}{\bar{u}^2} \text{ metres}, \tag{20}$$

with a in metres and $\bar{u}$ in metres per second. The latter approximation is based on the Reynolds number being around 10^4, and highlights the main dependence on a and $\bar{u}$. Substituting typical parameters $\bar{u} = 1\ \mathrm{m\,s^{-1}}$, $a = 0.1$ m and $\nu = 10^{-5}\ \mathrm{m^2\,s^{-1}}$ from §2.1, together with $\Delta\rho/\rho = 0.3$, we find

$$L = 160 \times 0.3 \times \left(\frac{0.1}{1.0}\right)^2 \times \left(\frac{0.1 \times 1.0}{10^{-5}}\right)^{1/4} \text{ metres} \approx 4.8 \text{ metres}. \tag{21}$$

For a worst case estimate, we take a wide pipe $a = 0.16$ m, a slow pumping velocity $\bar{u} = 0.24\ \mathrm{m\,s^{-1}}$, and a large density contrast $\Delta\rho/\rho = 0.7$, for which

$$L = 160 \times 0.7 \times \left(\frac{0.16}{0.24}\right)^2 \times \left(\frac{0.16 \times 0.24}{10^{-5}}\right)^{1/4} \text{ metres} \approx 390 \text{ metres}. \tag{22}$$

For a best case estimate, we take a narrow pipe $a = 0.05$ m, a fast pumping velocity $\bar{u} = 3.2\ \mathrm{m\,s^{-1}}$, and a small density contrast $\Delta\rho/\rho = 0.1$, for which

$$L = 160 \times 0.1 \times \left(\frac{0.05}{3.2}\right)^2 \times \left(\frac{0.05 \times 3.2}{10^{-5}}\right)^{1/4} \text{ metres} \approx 0.04 \text{ metres}. \tag{23}$$

Apart from the worst case estimate (22), these lengths are all much smaller than the total length of spacer or wash in the pipe, between 100 m and 2000 m.

The estimates are all proportional to $\nu^{-1/4}$ via Blasius' approximation (3). Thus using the highest spacer viscosity $\nu = 4 \times 10^{-4}\ \mathrm{m^2\,s^{-1}}$, or the lowest wash viscosity $\nu = 9 \times 10^{-7}\ \mathrm{m^2\,s^{-1}}$, instead of $\nu = 10^{-5}\ \mathrm{m\,s^{-1}}$, produces only about a factor of two variation either way in these lengths.

4.4 A variational principle for the Rayleigh number

We also considered a variational principle for estimating the critical Rayleigh number, which has the advantage of allowing for a spatially dependent eddy diffusivity $D(r)$. A solution of the eigenvalue problem for the critical Rayleigh number satisfies

$$0 = -Ra\, \hat{\mathbf{z}}\, C' - \nabla p' + \nabla\cdot(d\, \nabla \mathbf{u}), \tag{24a}$$
$$0 = -w + \nabla\cdot(d\, \nabla C'), \tag{24b}$$

where $d(\mathbf{x})$ is a position dependent dimensionless diffusivity. Multiplying (24b) by C', taking the inner product of (24b) with $\mathbf{u}$, and integrating by parts over the domain, we obtain

$$Ra\langle wC' \rangle = -\langle d\, |\nabla \mathbf{u}|^2 \rangle, \quad \langle wC' \rangle = -\langle d\, |\nabla C'|^2 \rangle, \tag{25}$$

where $\langle \cdot \rangle$ denotes a volume integral,

$$\langle \cdot \rangle = \int_0^{2\pi/k} \mathrm{d}z \int_0^1 r\mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta\ (\cdot). \tag{26}$$

The integration by parts involves discarding surface terms of the form $C'\frac{\partial C'}{\partial n}$, $\mathbf{u}{\cdot}\mathbf{n}p'$ and $\mathbf{u}\cdot\frac{\partial \mathbf{u}}{\partial n}$, which vanish on the pipe wall due to the boundary conditions of no flux, no penetration, and either no slip or free slip. The eigenmode is assumed to be periodic in z with period $2\pi/k$, so the contributions from the two end surfaces $z = 0$ and $z = 2\pi/k$ cancel.

Rearranging (25) we find [Proctor 1993],

$$Ra = \frac{\langle d\,|\nabla C'|^2\rangle\langle d\,|\nabla \mathbf{u}|^2\rangle}{\langle wC'\rangle^2} \geq \frac{\langle d\,|\nabla C'|^2\rangle\langle d\,|\nabla w|^2\rangle}{\langle wC'\rangle^2} \geq \frac{\langle d\,|\nabla_\perp C'|^2\rangle\langle d\,|\nabla_\perp w|^2\rangle}{\langle wC'\rangle^2} = S, \qquad (27)$$

where $\nabla_\perp = (\partial_x, \partial_y, 0)$ denotes the horizontal gradient operator. Proctor (1993) showed that the solution of the Euler–Lagrange equation for minimising S is a z-independent solution of the original eigenvalue problem (24a,b). The right hand inequality in (27) thus justifies the assumption made in §4.2 that the most unstable mode is independent of z.

For the uniform diffusivity case $d = 1$ we obtained the upper bounds $Ra_{\text{crit}} \leq 71.68$ and $Ra_{\text{crit}} \leq 11.51$, for rigid and free slip boundary conditions respectively, by considering the trial functions $C' = r - r^3/3$, and either $w = r - r^3$ or $w = r - r^3/3$. Taking $d = 1 - r$ as in (4), we obtain the bound $Ra_{\text{crit}} \leq 2.2$ for free slip boundary conditions respectively, which is in reasonable agreement with the value $Ra_{\text{crit}} \approx 1.84$ obtained by solving a discrete form of the eigenvalue problem numerically. Adopting this new value gives about a factor of six increase in the lengths of the mixed layers estimated in §4.3

5 Streamwise dispersion

For a typical well of length $L_{\text{p}} = 3000$ m, and a pumping speed $\bar{u} = 1\ \text{m}\,\text{s}^{-1}$, fluid is resident in the pipe for a time $t_{\text{res}} = 3000$ s. During this time the mixed layer between pure wash and pure space will tend to broaden due to turbulent mixing, and also due to Taylor dispersion [Taylor 1953], the effective streamwise diffusion due to variations in streamwise velocity across the pipe,

Taylor (1954a) proposed the value $\bar{D} = 10.1au^\star$ for the effective streamwise diffusivity of a passive tracer in turbulent pipe flow, based on the 'universal' model outlined in §2.2. Experimental evidence in support of this value may be found in Taylor (1954a,b). This value is comparable with the one obtained by substituting the eddy viscosity $D = kau^\star$ into the Taylor dispersion formula $\bar{D} = a^2\bar{u}^2/48\nu$ [Taylor 1954b] for laminar pipe flow with molecular viscosity ν, which leads to $\bar{D} = 1.57au^\star Re^{1/4}$ with the aid of Blasius' approximation (3).

An order of magnitude estimate for the spreading of the mixed layer due to Taylor dispersion is

$$L_{\text{Taylor}} = \sqrt{2\bar{D}t_{\text{res}}} = \sqrt{20.2au^\star L_{\text{p}}/\bar{u}} = 1.92\sqrt{aL_{\text{p}}}Re^{-1/16} \approx \sqrt{aL_{\text{p}}}, \qquad (28)$$

where the residence time $t_{\text{res}} = L_{\text{p}}/\bar{u}$. Blasius' formula (3) has been used for the ratio between the friction velocity $u^\star$ and the mean pumping velocity $\bar{u}$. For the three cases considered in §4.3 the Reynolds number lies in the range $3800 \leq Re \leq 16000$, for which $1.92Re^{-1/16} \approx 1$ to within 10%.

In other words, the mixed layer grows to a length which is roughly the geometrical mean of the pipe radius and the pipe length, with a very weak dependence on the pumping velocity and material properties. Pumping faster leads to a shorter residence time which almost exactly compensates for the larger effective diffusivity. For the three cases considered in §4.3, and a worst case 5000 m pipe, the estimated lengths after Taylor dispersion are 22 m, 28 m, and 16 m respectively. Again, these lengths are all somewhat smaller than the total length of spacer or wash in the pipe, which is between 100 m and 2000 m.

Taylor (1954a) reported some experiments which suggest that the dispersion $\bar{D}$ may be up to a factor of two larger in a slightly curved pipe, even one where the radius of curvature is a hundred times the pipe radius. This would increase the above estimates by $\sqrt{2}$.

We also considered some more sophisticated one-dimensional models of two phase flow based upon averaging over the pipe's cross-section. These models all reduced to advection-diffusion equations for the volume fraction of spacer, or of particles, where the advection velocity is comparable to the pumping velocity $\bar{u}$ and the diffusivity is comparable with the turbulent diffusivity D. Thus the above conclusion based on dimensional analysis seemed robust — the streamwise diffusivity is too small to allow substantial broadening of the mixed layer before it reaches the end of the pipe.

6 Effects of a halt in pumping

The above estimates are all based upon an effective turbulent diffusivity D many times larger than either the molecular viscosity or the molecular diffusivity of particles. This large diffusivity is maintained by wall turbulence generated by the pumping velocity $\bar{u}$. If pumping were to cease, the diffusivities appearing in (10) would return to their molecular values, leading to a large increase in the maximum stable density gradient calculated in §4.3. Thus the mixed layer, which had previously been stabilised by turbulent diffusion, would be become unstable and start to grow. However, the growth of an instability would itself drive turbulence which would help to stabilise the layer. To obtain a 'worst case' estimate, we supposed that the turbulence driven by pumping would decay quickly, and tried to estimate how rapidly the mixed layer would spread in its the absence.

We tried to model the resulting turbulent convection by assuming that the spanwise mixing model in §3 only "sees" the effective density contrast over a distance comparable with the pipe radius. In other words, the effective density contrast is $a\Delta\rho/L$ instead of $\Delta\rho$. Noting, from (6) and (3), that the finger velocity u_{f} is approximately proportional to $\Delta\rho^{1/2}$, we first considered a simple model in which the mixed layer grew with this modified finger velocity,

$$\frac{\mathrm{d}L}{\mathrm{d}t} = u_{\mathrm{f}}\sqrt{\frac{a}{L}} \quad \Rightarrow \quad L^{3/2} = L_0^{3/2} + tu_{\mathrm{f}}\sqrt{a}. \tag{29}$$

Here u_{f} is the finger velocity calculated in §3, and $u_{\mathrm{f}}\sqrt{a/L}$ the modified value based on a reduced density contrast. Taking typical values $u_{\mathrm{f}} = 10\ \mathrm{m\,s^{-1}}$, $a = 0.1$ m, $L_0 = 30$ m, we find that L has doubled after 50 s. It seems that the mixed layer will not grow substantially so long as the pumping is not halted for more than a few seconds.

6.1 A nonlinear diffusion model

A more sophisticated approach is to replace $\Delta\rho$ by $a\partial\rho/\partial z$ in equation (6) for the friction velocity,

$$u^\star = \sqrt{\sigma_{\mathrm{wall}}/\rho} = a\left(\frac{g}{\rho}\frac{\partial\rho}{\partial z}\right)^{1/2}. \tag{30}$$

As above, this replacement is motivated by a mixing length approach, in which the effective density contrast is that seen over a mixing length of approximately one pipe radius. In the absence of a mean flow due to pumping, we expect the concentration C of the dense component to diffuse with the eddy diffusivity $D = kau^\star$. This leads to a nonlinear diffusion equation for the concentration C,

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial z}\left(D\frac{\partial C}{\partial z}\right) = ka^2\left(\alpha\frac{g}{\rho}\right)^{1/2}\frac{\partial}{\partial z}\left(\frac{\partial C}{\partial z}\right)^{3/2}, \tag{31}$$

where α is the expansion coefficient, $\rho = \rho_0 + \alpha C$, as in §4, and k is von Karman's constant.

Holmes *et al.* (1991) and Baird *et al.* (1992) performed experiments on turbulent convective mixing between salty and fresh water in tubes. Their experimental setup was quite similar to the problem considered here, except their tubes were somewhat narrower ($a < 3.3$ cm) and much shorter (less than 1.3 m). They proposed an effective diffusivity of the same form,

$$D = \ell^2 \left(\frac{g}{\rho} \frac{\partial \rho}{\partial z} \right)^{1/2}, \tag{32}$$

based on dimensional analysis and an empirical study of their experimental data. The parameter ℓ is a turbulent mixing length, which they found to be somewhere between their pipe radius and pipe diameter by fitting their experimental concentration profiles.

6.2 Diffusion timescales

The change of variables $C(z,t) = (\Delta\rho/\alpha)\, c(z/L, t/T)$ puts the nonlinear diffusion equation (31) into the dimensionless form

$$\frac{\partial c}{\partial \tau} = \frac{\partial}{\partial x} \left(\frac{\partial c}{\partial x} \right)^{3/2}, \tag{33}$$

where $x = z/L$ and $\tau = t/T$ are dimensionless independent variables. The diffusion timescale T is given by

$$T = \frac{2}{3k} \left(\frac{L}{a} \right)^2 \left(\frac{\rho L}{g \Delta\rho} \right)^{1/2}. \tag{34}$$

Eliminating the lengthscale L using the mixed length estimate (20) from §4.3, the diffusion timescale T may be rewritten as

$$T = \frac{2048}{3k} \left(\frac{\Delta\rho}{\rho} \right)^2 \frac{g^2 a^3}{\bar{u}^5} \left(\frac{\bar{u} a}{\nu} \right)^{5/8}. \tag{35}$$

We found in §4.3 that L varied by four orders of magnitude between the best and worst cases considered. Since the diffusion timescale T is roughly proportional to L^2, we find an even more sensitive dependence of T upon the problem parameters, in particular a, $\bar{u}$ and $\Delta\rho/\rho$. For the three cases considered in §4.3, we find $T \approx 1200$ s, $T \approx 7 \times 10^7$ s, and $T \approx 0.2$ s respectively. For a more plausible estimate in the last case, corresponding to (23), if we take the length $L = 16$ m based on Taylor dispersion from §5 the timescale becomes $T \approx 4 \times 10^5$ s.

Some neglected constants may change these estimates by a factor of three or four. For instance, the experiments of Holmes *et al.* (1991) and Baird *et al.* (1992) suggest that the mixing length ℓ should be closer to $2a$ than to a, which would reduce the timescales in (34) and (35) by a factor of four. In the lower stages of the pipe where the pipe axis is some way from vertical g should perhaps be replaced by the reduced gravity $g \cos\beta$, though it is not clear what effect a slanted pipe would have on the mixing length. A numerical solution of (33) with initial conditions

$$c(x,0) = \begin{cases} 0 & \text{if } x < -1, \\ x + \dfrac{1}{2} & \text{if } |x| < 1, \\ 1 & \text{if } x > 1, \end{cases} \tag{36}$$

shows that the time to double the width of the profile (*i.e.* halve the maximum gradient) is only $0.3T$. Conversely, the doubling time is $2^{5/2}T \approx 5.7T$ for the similarity solution derived below.

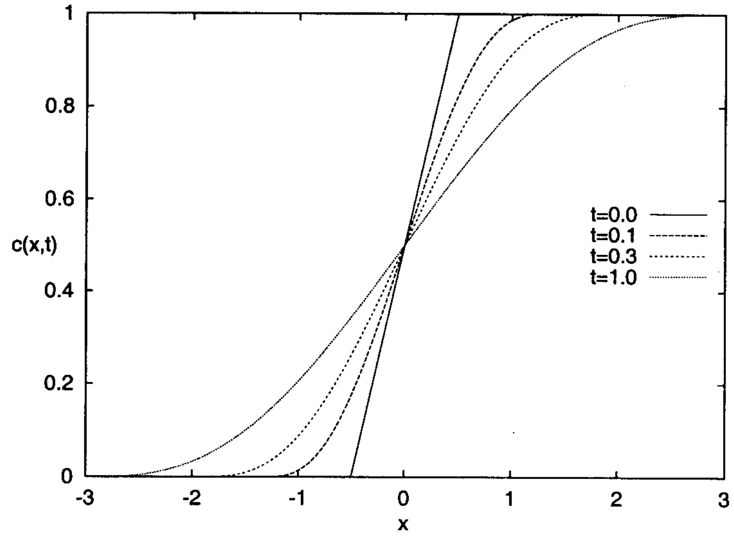
However, despite all these possible modifications the timescales in (34) and (35) for realistic initial mixed layer thicknesses all suggest that the mixed layer would not grow substantially if pumping were halted for a few seconds or even a few minutes.

6.3 Approach to a similarity solution

Equation (33) has a similarity solution of the form $c(x,\tau) = f(x/\tau^{2/5})$, where

$$f(\eta) = \begin{cases} 0 & \text{if } \eta < -\eta_0, \\ (3\eta^5 - 10\eta^3\eta_0^2 + 15\eta\eta_0^4)/3375 + \dfrac{1}{2} & \text{if } |\eta| < \eta_0, \\ 1 & \text{if } \eta > \eta_0. \end{cases} \tag{37}$$

Figure 3: Numerical solution of the nonlinear diffusion equation (33) with initial conditions (36) computed using the NAg routine D03PGF. The solutions at $t = 0.3$ and $t = 1$ are indistinguishable from the similarity solution (37).



The constant $\eta_0 = 15^{3/5}2^{-4/5} \approx 2.916$ is determined by the requirement that $f(\eta) = 0, 1$ at the two turning points $\eta = \pm\eta_0$. As with the more familiar similarity solution to the porous medium equation, deviations from the background states $c = 0, 1$ are confined to a compactly supported region, in this case $|\eta| < \eta_0$. However, unlike the similarity solution to the porous medium equation, f, f' and f'' are all continuous at $\eta = \pm\eta_0$. Figure 3 shows a numerical solution of (33) with piecewise linear initial conditions (36) approaching the similarity solution. By $t = 1$ the numerical solution is indistinguishable from the similarity solution with the time origin offset by 0.04.

Equation (33) may also be rewritten as a porous medium (Barenblatt) equation for the density gradient h,

$$\frac{\partial h}{\partial \tau} = \frac{\partial^2}{\partial x^2}\left(h^{3/2}\right) = \frac{3}{2}\frac{\partial}{\partial x}\left(h^{1/2}\frac{\partial h}{\partial x}\right), \text{ where } h = \frac{\partial c}{\partial x}. \tag{38}$$

The above similarity solution then corresponds to a solution of (38) forced by the boundary condition $h = h_0\tau^{-2/5}$ at $x = 0$, and satisfying $h \to 0$ as $x \to \pm\infty$. The constants η_0 and h_0 are related by $\eta_0^4 = 225h_0$.

7 Conclusions

We found in §4.3 that the length of the initial mixed layer is proportional to the density contrast $\Delta\rho/\rho$, proportional to the pipe radius squared, and approximately *inversely* proportional to the pumping velocity squared. Pumping faster causes more turbulence and so, somewhat counterintuitively, allows *less* mixing between the wash and the spacer. This favours a narrow pipe and a high pumping velocity. A small density contrast is also helpful.

The subsequent growth of the mixed layer due to Taylor dispersion was found in §5 to be proportional to the geometric mean of the pipe radius and pipe length, but almost independent of the pumping rate. Pumping rapidly produces enhanced turbulent dispersion which almost exactly compensates for the reduced residence time. For two of the three cases considered, Taylor dispersion was found to be the dominant spreading mechanism. This also favours a narrow pipe.

Halting pumping for a few seconds or even a few minutes seems unlikely to cause substantial growth of the mixed layer, but again the growth rate would be minimised by a narrow pipe and a high pumping velocity.

The critical Rayleigh number and the Taylor dispersion coefficient are both sensitive to the mean flow profile, which could presumably be modified by changing the rheology of the spacer fluid.

8 Contributors

The problem was posed by Ian Frigaard and Guiliano Sona. Academic contributors included P.J. Dellar, G. Duursma, A. Fitt, E.J. Hinch, O.G. Harlen, J. King, J.R. Ockendon, D. Parker, C. Please, N. Stokes and S. Wilson. The report was prepared by P.J. Dellar. Figures (1) and 2(a) were provided by Ian Frigaard.

9 References

BAIRD, M.H.I., ARAVAMUDAN, K., RAMA RAO, N.V., CHADAM, J. & PEIRCE, A.P., (1992) Unsteady axial mixing by natural convection in a vertical column, *AIChE J.,* **38**, 1825–1834.

BATCHELOR, G.K. & NITSCHE, J.M., (1993) Instability of stratified fluid in a vertical cylinder, *J. Fluid Mech.,* **252**, 419–448.

CHANDRASEKHAR, S., (1961) Hydrodynamic and Hydromagnetic Stability, *Oxford University Press (Dover reprint 1981).*

DRAZIN, P.G. & REID, W.H., (1981) Hydrodynamic Stability, *Cambridge University Press.*

GOLDSTEIN, S. (ed), (1938) Modern Developments in Fluid Dynamics, *Oxford University Press.*

HOLMES, T.L., KARR, A.E. & BAIRD, M.H.I., (1991) Effect of unfavorable continuous phase density gradient on axial mixing, *AIChE J.,* **37**, 360–366.

LOWELL, M.E. & ANDERSON, J.L., (1982) Stable concentration gradients in a vertical tube, *Chem. Eng. Commun.,* **18**, 93–96.

NELSON, E.B. (ed), (1990) Well Cementing, *Elsevier: Amsterdam.*

PROCTOR, M.R.E., (1993) Proof that $\kappa = 0$ gives the minimum critical Rayleigh number in the case of constant density gradient, *J. Fluid Mech.,* **252**, 447–448. (Appears as an appendix to Batchelor & Nitsche (1993).)

TAYLOR, G.I., (1953) Dispersion of soluble matter in solvent flowing slowly through a tube, *Proc. Roy. Soc. Lond.,* A**219**, 186–203. Also in *Scientific Papers of G. I. Taylor, vol. IV*, pp 225–243, ed. G. K. BATCHELOR, Cambridge University Press.

TAYLOR, G.I., (1954a) Dispersion of matter in turbulent flow through a pipe, *Proc. Roy. Soc. Lond.,* A**223**, 446–468. Also in *Scientific Papers of G. I. Taylor, vol. II*, pp 466–488, ed. G. K. BATCHELOR, Cambridge University Press.

TAYLOR, G.I., (1954b) Diffusion and mass transport in tubes, *Proc. Phys. Soc. Lond.,* B**67**, 857–869.

TOWNSEND, A.A., (1976) The Structure of Turbulent Shear Flow, *Cambridge University Press.*

YIH, C.-S., (1959) Thermal instability of viscous fluids, *Quart. Appl. Maths.,* **17**, 25–42.