4 Some studies on the deformation of the membrane in an RF MEMS switch

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Abstract

Radio Frequency (RF) switches of Micro Electro Mechanical Systems (MEMS) are appealing to the mobile industry because of their energy efficiency and ability to accommodate more frequency bands. However, the electromechanical coupling of the electrical circuit to the mechanical components in RF MEMS switches is not fully understood.

In this paper, we consider the problem of mechanical deformation of electrodes in RF MEMS switch due to the electrostatic forces caused by the difference in voltage between the electrodes. It is known from previous studies of this problem, that the solution exhibits multiple deformation states for a given electrostatic force. Subsequently, the capacity of the switch that depends on the deformation of electrodes displays a hysteresis behaviour against the voltage in the switch.

We investigate the present problem along two lines of attack. First, we solve for the deformation states of electrodes using numerical methods such as finite difference and shooting methods. Subsequently, a relationship between capacity and voltage of the RF MEMS switch is constructed. The solutions obtained are exemplified using the continuation and bifurcation package AUTO. Second, we focus on the analytical methods for a simplified version of the

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problem and on the stability analysis for the solutions of deformation states. The stability analysis shows that there exists a continuous path of equilibrium deformation states between the open and closed state.

4.1 Introduction

Radio Frequency switches (RF) of Micro Electro Mechanical Systems (MEMS) have achieved considerable attention in the mobile industry because of the need for an increase in frequency bands and energy efficiency. RF MEMS switches have several advantages over traditional semiconductors such as power consumption, lower insertion loss, higher isolation and good linearity. However, a thorough understanding of the electromechanical coupling between the electrical circuit and mechanical component of an RF MEMS switch is not fully established and this forms the subject of the present paper.

Problem description: RF MEMS switches typically consist of two electrodes which are thin membranes parallel to each other as shown in the Figure 4.1. In the schematic cross-section of the switch, Figure 4.1(a), the thick black lines indicate the bottom and top electrodes in which the bottom electrode is fixed and the top electrode is free to deform with its ends fixed. In the presence of equal and opposite electric charge Q in the electrodes, the top electrode deforms to balance the electrostatic force $F_{\text{electrostatic}}$ induced with its mechanical spring force F_{spring} for equilibrium. To avoid the contact between the two electrodes, a dielectric of thickness d_{diel} is provided on the top of the bottom electrode as indicated with dashed lines in Figure 4.1(a). Further, the thickness of the top electrode is h and it is separated by a distance g from the dielectric in the unforced state. The deformed shape of the top electrode at equilibrium is described by the displacement $u(x)$.

The equilibrium states are the critical points at which the the total energy is minimized. The total energy E_{tot} is given by the sum of the electrical energy E_{el} and the mechanical energy E_{mech}:

$$E_{\text{tot}} = E_{\text{el}} + E_{\text{mech}}.$$

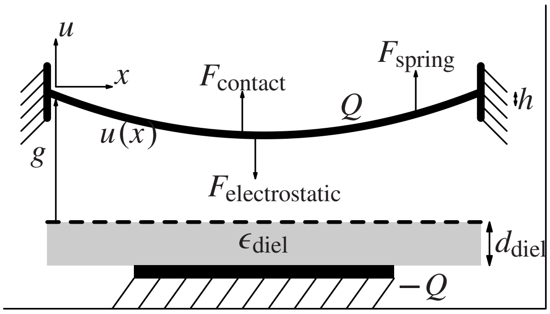
The electrical energy E_{el} is given as

$$E_{\text{el}} = \frac{Q^2}{2C} \quad \text{with} \quad C(u(x,y)) := \int_{A_{\text{bot}}} \frac{\epsilon_0 \, \mathrm{d}x\mathrm{d}y}{g + u(x,y) + d_{\text{diel}}/\epsilon_{\text{diel}}},$$

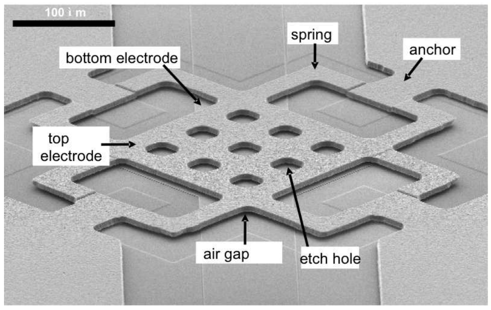
where C is the capacitance, Q the electric charge, u(x,y) the displacement, ϵ_0 the vacuum permittivity coefficient, d_{diel} the thickness of dielectric, ϵ_{diel} the dielectric constant and A_{bot} the area of bottom electrode. In determining the capacitance C, the two electrodes are assumed to be parallel under no charge in the unforced state. Taking only the bending forces into account and assuming the thickness of the electrode to be very small with zero initial stress, the mechanical energy is given by

$$E_{\text{mech}} = \int_{A_{\text{top}}} \frac{D}{2} |\Delta u|^2 \, \mathrm{d}x\mathrm{d}y, \quad \text{where} \quad D = \frac{2h^3 Y}{3(1-\nu^2)},$$

h is the thickness, A_{top} area of the top electrode, Y is Young's modulus and ν is the Poisson ratio of the top electrode.



(a)



(b)

Figure 4.1: (a) Schematic cross-section of a capacitive RF MEMS switch. (b) Scanned electron microscope picture of a capacitive RF MEMS switch.

Problem formulation: The main problem is the following: find all the displacement states i of the top electrode $u_{\text{eq},Q,i}(x, y)$ for which the forces on the top electrode are in equilibrium at a fixed charge Q on the top electrode (or for a fixed voltage V between the electrodes). Several sub-problems are posed as follows:

- Is there always a continuous path of equilibrium states $u_{\text{eq},Q,i}(x, y)$ between the open state $u_{\text{eq},Q,i} = 0$ for all $x, y \in A_{\text{top}}$ and the closed state $u_{\text{eq},\infty,N} = -g$ for all $x, y \in A_{\text{bot}}$.
- Is there a function $f(u_{\text{eq},Q,i}(x, y), Q)$ that is monotonically increasing along this path?
- Can it be shown that along this continuous path $dE_{\text{mech}}dC > 0$ is always valid? Here E_{mech} is the mechanical energy and C is its capacitance.
- Is there a simple way to determine whether a state is stable or unstable at a fixed voltage or charge?
- For which geometries and boundary conditions is the problem analytically solvable? Most interesting is the situation in which the top electrode springs are clamped (zero displacement and zero slope) at some points of its boundary.

- The dynamics of the structure under the presence of gas damping is a related interesting problem.

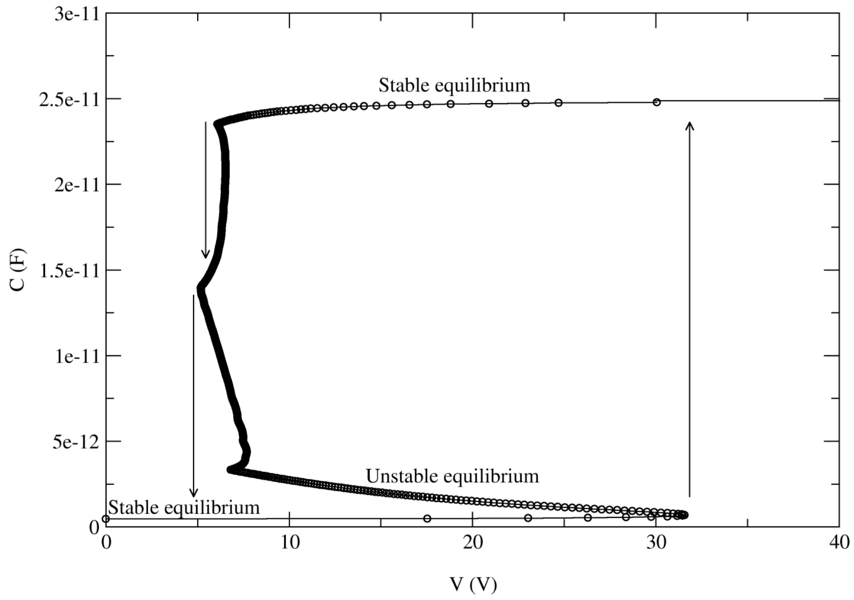
Finite element method: The deformation shapes at equilibrium are often solved using finite element packages. However, it is not straightforward to find multiple or all deformation shapes at equilibrium for a given voltage V as some of them are unstable. Given the deformation shape of the electrode $u(x, y)$, the capacitance of the RF MEMS switch is determined. Such a CV-curve is shown for two examples of RF MEMS switch in Figure 4.2(a) and (b). Multiple values of capacitance C for a given voltage V are clearly seen in Figure 4.2; a phenomenon called *hysteresis*.

Overview: The equilibrium problem of a RF MEMS switch is interesting both from a practical as well as a mathematical point of view. It should be stressed, however, that the entire problem is too general and difficult. Hence, in the present paper, we have considered a one dimensional version to obtain some interesting insights and solutions.

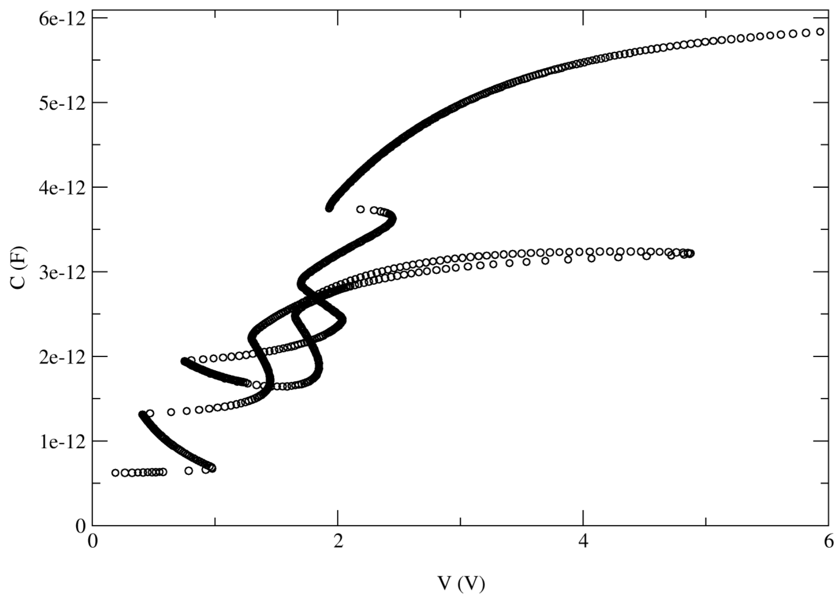
First, we prove that under certain conditions on the total energy of RF MEMS, the deformation states at equilibrium are stable. Second, we formulate an inequality from which the stability conditions are derived. Third, we prove that when the top electrode touches the dielectric, its deformation shape will have no gaps in the contact area with dielectric. Finally, we prove the existence of a continuous path of equilibrium states under some given mild conditions on the energy of the system.

Besides these theoretical results, we make use of numerical methods such as the finite difference and shooting methods to solve for the displacements of the deformation shape of the top electrode. To acquire insight into the nature of solutions, we generate several sets of deformation shapes using the continuation and bifurcation package AUTO. AUTO [3] typically generates sets of solutions to a given problem by continuation, i.e., it calculates a solution for any given parameter of the system. The main advantage of this approach as opposed to using finite element packages is that the non-unique or multiple solutions for a given problem are easily found. In addition, an article on modeling MEMS by using continuation is in preparation (see [14]).

The paper is divided into two parts. In the first part, we present the numerical methods to the present problem to gain some insight into the nature of solution. We then employ the continuation method AUTO and a shooting method to generate numerical solutions. In the second part, we discuss various analytical approaches to the problem. We derive full solutions to the linearized problem. Linear problems with any suitable boundary conditions have a unique solution and hence, no hysteresis is found. Finally, we present various other results for the nonlinear problem.



(a)



(b)

Figure 4.2: Calculated CV-curve (capacitance-voltage characteristic) of two different switches. (a) CV-curve of the switch of Figure 4.1. (b) CV-curve of the so-called "seesaw" RF MEMS switch.

4.2 Numerical Methods

4.2.1 Finite difference scheme

We consider a one dimensional model problem of the RF MEMS switch which exhibits the important qualitative aspects of the system and its non-dimensional form follows from the minimization of total energy:

$$\frac{\partial^4 u}{\partial x^4} = -\frac{\epsilon_0 V^2}{1-\eta+u} + \phi(u) \quad \text{on } x \in [0,1] \tag{4.1}$$

$$\text{with } u = \frac{\partial u}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = 1,$$

where $u(x)$ is the displacement, η a small parameter, ϵ_0 the vacuum permittivity, V the voltage between the electrodes and $\phi(u)$ the contact force between the plate and the dielectric which is non zero for $u < -1$, i.e., when the scaled downward displacement is greater than the scaled gap $g = 1$ between the electrodes.

A simple finite difference scheme for the 1D model problem (4.1) is developed and implemented in MATLAB. The numerical solutions of this scheme are compared to the analytical approximations and they can serve as a basis for more advanced 2D simulations in the future. To obtain the finite difference scheme, we first divide the domain into $n-1$ grid cells with grid size Δx and n grid points. The displacement at each grid point x_i is denoted as $u(x_i) = u_i$. The biharmonic operator in (4.1) is discretized using a central difference scheme as follows:

$$\frac{\partial^4 u}{\partial x^4} \approx \frac{u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2}}{\Delta x^4} + \mathcal{O}(\Delta x^2) \quad i = 2, \ldots, n-1. \tag{4.2}$$

Near the boundaries, we employ the boundary conditions $u_1 = u_n = 0$, $u_2 - u_0 = 0$ and $u_{n+1} - u_{n-1} = 0$ which are second order central difference approximations to the boundary conditions in order to get a consistent approximation. Substituting the approximation of biharmonic operator (4.2) in (4.1), the finite difference discretization takes the following form:

$$A\mathbf{u} = -\frac{\epsilon_0 V^2}{1-\eta+\mathbf{u}} + \phi(\mathbf{u}), \tag{4.3}$$

where A is a constant matrix and $\mathbf{u}$ is the displacement vector at the points $\mathbf{x} = x_i, i = 2, \ldots, n-1$. The discretized biharmonic operator A can be efficiently inverted using an iterative solver such as conjugate gradient method (CG). However, the right hand side of the equation is non-linear and hence, it is typically treated with a fixed-point iteration. The fixed-point iteration scheme is easily described by rewriting (4.3) as follows:

$$\mathbf{u}^{k+1} = A^{-1}\left(-\frac{\epsilon_0 V^2}{1-\eta+\mathbf{u}^k} + \phi(\mathbf{u}^k)\right). \tag{4.4}$$

Now given a guess $\mathbf{u}^k$ for displacement $\mathbf{u}$, we compute for displacement $\mathbf{u}^{k+1}$ using (4.4) per grid cell and iterate with respect to k until the solution converges. From a physical point of view, it is clear that the equation is not uniquely solvable for a certain range of voltages V. In fact, this is reflected in the fixed-point iteration scheme as it could converge to two different solutions for the displacement vector. Typically, the solution to which it converges depends on the starting point for the iteration. This suggests that a CV-curve with stable solutions of the system can be drawn. To draw the CV-curve, we start with a low voltage V for which the solution is unique and stable. Subsequently, we increment voltage V and use the previous solution as the starting for the fixed-point iteration scheme which resulted in a quick convergence to the nearby solution. Similarly, to obtain the remaining branch of solutions, we started with a high voltage V and repeated the previous procedure by decreasing the voltage V. This has lead us to construct a "continuous" branch of the CV-curve.

4.2.2 Shooting method

In this section, we consider a shooting method to solve the nonlinear one dimensional model problem of RF MEMS switch. The shooting method in some sense is the easiest method to find numerical solutions for a boundary value problem of a nonlinear ordinary differential equation. It relaxes the problem by ignoring one of the boundary conditions and replacing it by a "free" initial choice instead. This initial choice is adapted until the obtained solution satisfies the boundary condition that was ignored. We refer to [11] for a detailed description of the shooting method.

We distinguish three situations for the shooting method:

1. The top electrode touches the dielectric over some interval.
2. The top electrode touches the dielectric at one point.
3. The top electrode does not touch the dielectric.

Each of these cases contribute to different parts of the CV-curve. We describe the shooting method in detail for the first situation, i.e., when the plate touches the dielectric on some interval, and solve the shooting problem. The remaining two situations are solved analogously and hence, we omit the description. Finally we compute the CV-curve according to

$$C(v) = \frac{\Lambda\epsilon_0}{g}\int_{-1}^{1}\frac{dx}{1+u(x;v)-\eta}. \tag{4.5}$$

For all computations, we employ MATHEMATICA 6.

Electrode touches dielectric over some interval

Because of symmetry, we consider the electrode membrane in the half interval [0, 1] and take that the membrane touches the dielectric at $x = a$, where a is the distance measured from the fixed end $x = 0$ of the membrane and $0 \leq a < 1$. The nonlinear differential equation describing the shape of the membrane $u(x)$ between the fixed end and the contact with the dielectric is

$$\frac{\partial^4 u}{\partial x^4} = u'''' = -\frac{\epsilon_0 V^2}{1 - \eta + u}, \tag{4.6}$$

with boundary conditions

$$u(1) = 0, \quad u'(1) = 0, \quad u(a) = -1, \quad u'(a) = 0, \quad u''(a) = 0. \tag{4.7}$$

Here, an additional condition $u(a) = -1$ is required for the unknown contact point at $x = a$ on the dielectric.

It is convenient to make a change of variable x to $\tilde{x} = x - a$, $\tilde{u}(\tilde{x}) = u(x)$. Consequently, boundary conditions (4.7) now become as

$$\tilde{u}(1-a) = 0, \quad \tilde{u}'(1-a) = 0, \quad \tilde{u}(0) = -1, \quad \tilde{u}'(0) = 0, \quad \tilde{u}''(0) = 0, \tag{4.8}$$

and (4.6) remains the same as

$$\tilde{u}'''' = -\frac{\epsilon_0 V^2}{1 - \eta + \tilde{u}}. \tag{4.9}$$

In order to solve (4.6) and (4.7), we study the initial value problem for (4.9) with initial conditions

$$\tilde{u}(0) = -1, \quad \tilde{u}'(0) = 0, \quad \tilde{u}''(0) = 0, \quad \tilde{u}'''(0) = P, \tag{4.10}$$

which has a solution $\tilde{u}(\tilde{x}; P)$ with P an unknown parameter to be found later. Now, it remains to find a solution $P = P_s$ such that the solution of (4.9) and (4.10) satisfies the following condition at some point $b > 0$:

$$\tilde{u}(b; P_s) = 0, \quad \tilde{u}'(b; P_s) = 0. \tag{4.11}$$

Setting $a = 1 - b$, we obtain the solution $u(x) = \tilde{u}(\tilde{x}; P_s)$ satisfying (4.6) and (4.7). Note that, for the case $b > 1$ a solution of (4.6) and (4.7) does not exist.

The function $\tilde{u}(\tilde{x}; P)$ increases as function of P, see Figure 4.3(a). For small P, $\tilde{u}(\tilde{x}; P)$, as a function of $\tilde{x}$, increases, reaches a negative maximum and then decreases, see curves below the red one in Figure 4.3(a). For larger P, $\tilde{u}(\tilde{x}; P)$ increases and has positive first derivative where it crosses the line $\tilde{u} = 0$ for the first time, see curves above the red one in Figure 4.3(a). For $P = P_s$ the function $\tilde{u}(\tilde{x}; P)$ has a local maximum $\tilde{u} = 0$ (the red curve in Figure 4.3(a). This function satisfies the conditions (4.11) and b is the value of $\tilde{x}$ at which $\tilde{u}$ has the local maximum.

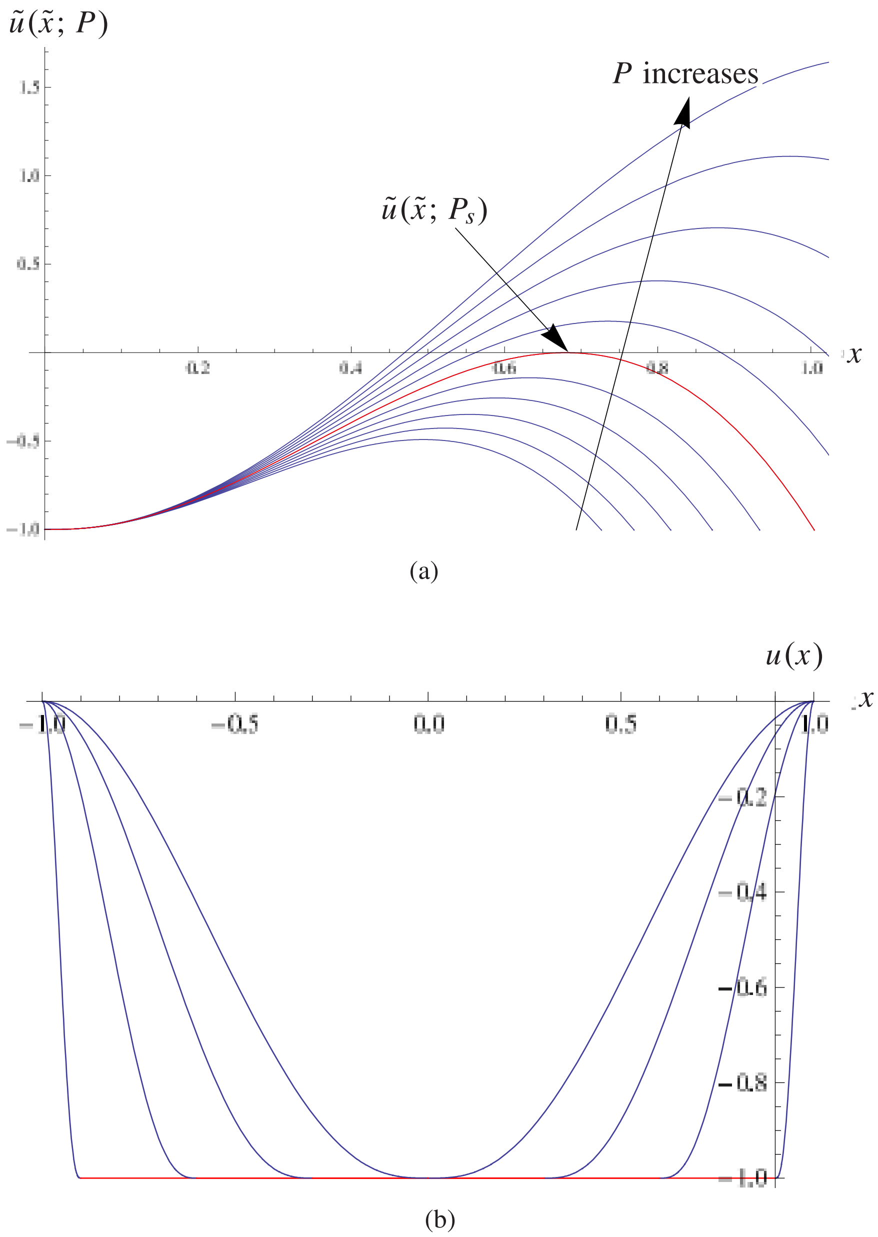


Figure 4.3: (a) The function $\tilde{u}(\tilde{x};\, P)$ for different values of the shooting parameter P and $V = 890$. Here, $\tilde{u}(\tilde{x};\, P)$ increases as P increases. The red curve corresponds to a solution $\tilde{u}(\tilde{x};\, P_s)$ which satisfies (4.11) and solves (4.6) and (4.7). (b) The membrane shape for different values of V. The red line depicts a part of membrane in contact with dielectric. The blue curve is the shape of the membrane between the support and the dielectric.

Next, we present an alternative method for solving (4.6) and (4.7). This method is convenient for fast construction of a CV-curve because it requires solving a boundary value problem only once. Then using a scaling argument we get an easily calculable expression for C.

First we rescale $\tilde{x}$ according to $\hat{x} = \tilde{x}/(1-a)$ and $\hat{u}(\hat{x}) = \tilde{u}(\tilde{x})$. Then the boundary value problem (4.6) and (4.8) becomes

$$\hat{u}'''' = -\frac{\epsilon_0 V^2 (1-a)^4}{1-\eta+\hat{u}}, \tag{4.12}$$

$$\hat{u}(1) = 0, \quad \hat{u}'(1) = 0, \quad \hat{u}(0) = -1, \quad \hat{u}'(0) = 0, \quad \hat{u}''(0) = 0. \tag{4.13}$$

To solve (4.12) and (4.13) using the shooting method routine implemented in MATHEMATICA 6 we rewrite (4.12) as follows

$$\hat{u}''''(\hat{x}) = -\frac{\epsilon_0 \hat{V}(\hat{x})^2}{1-\eta+\hat{u}(\hat{x})}, \quad \hat{V}'(\hat{x}) = 0. \tag{4.14}$$

Here the unknown $V^2(1-a)^4$ is described as an unknown constant function $\hat{V}(\hat{x})$. A solution $\hat{u}(\hat{x})$ and $\hat{V}(\hat{x}) = V_s$ of (4.14) describes the shape of the membrane $u(x) = \hat{u}(x)$ for $a = 0$, and V_s is the minimum value of V for which (4.6) and (4.7) has a solution. A solution $u(x)$ for arbitrary $V > V_s$ is written as

$$u(x) = \hat{u}((x-a)/(1-a)), \quad a = 1 - \sqrt{\frac{V_s}{V}}.$$

The shape of the membrane is

$$u(x) = \begin{cases} \hat{u}((|x|-a)/(1-a)), & \text{for } a < |x| \le 1, \\ -1, & \text{for } |x| \le a, \end{cases}$$

see Figure 4.3(b).

With increasing V the contact with the dielectric increases and the membrane shape between the support and the dielectric becomes steeper.

The value of C is computed from (4.5) as

$$C(V) = \frac{2\Lambda\epsilon_0}{g}\left(\frac{1-\sqrt{V_s/V}}{\eta} + \sqrt{\frac{V_s}{V}} I_1\right), \quad \text{where} \quad I_1 = \int_0^1 \frac{d\hat{x}}{1+\hat{u}(\hat{x})+\eta},$$

from which follows that $C(V)$ has a horizontal asymptotic

$$\lim_{v\to\infty} C(V) = \frac{2\Lambda\epsilon_0}{g\eta}. \tag{4.15}$$

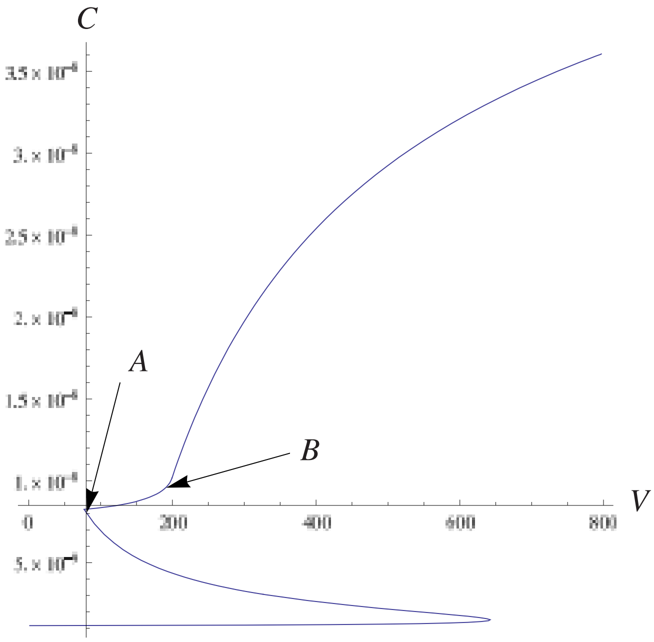


Figure 4.4: CV-curve for all three cases. The membrane first touches the dielectric at the point A. The change between the situations when the membrane touches the dielectric at one point, and on some interval is indicated by the point B.

CV-curve and influence of model parameters

Summarizing the results of the CV-curves for all three situations, we construct the CV-curve for all V, see Figure 4.4. The complete CV-curve has discontinuous derivative at the transition point when the membrane touches the dielectric for the first time (point A in Figure 4.4). At the transition between the situations when the membrane touches the dielectric at one point and on some interval (point B in Figure 4.4), the CV-curve is C^1. For some interval of V three values of C are possible (see Figure 4.4). This is a consequence of the non-uniqueness of the solution to the original problem for u.

4.3 The continuation problem

AUTO is a software package that is used for finding and displaying solutions, and tracking bifurcations of solutions of ordinary differential equations (ODEs) by continuation of some system parameter.[8] A bifurcation is, loosely formulated, a sudden change in the qualitative behaviour of ODEs when some system parameter (or *bifurcation parameter*) crosses a certain threshold (the *critical value*). For example,

[8] The package has been developed initially by E. Doedel and subsequently expanded by a range of authors, see [3]

an equilibrium solution may loose stability when the bifurcation parameter crosses a critical value. For more on the notion of bifurcation, see [6].

By continuation we mean the process of changing this system parameter and calculating the deformation of a solution when this parameter is changed. A typical continuation starts out with some (acquired) solution for the system with a certain value for the system parameter. Then the parameter is changed, and the solution is calculated for each value of the parameter. AUTO also detects bifurcations when they take place. So, in order to do a continuation, one has to find one solution for a specific value of the bifurcation parameter (often zero is a smart choice). By changing a parameter (i.e. by a *continuation* in one of the parameters) the solution generically changes as well. This solution can be found by AUTO, for each value of the bifurcation parameter.

Most continuation software, and especially AUTO, allows for continuation in two or more parameters as well. AUTO is not only able to perform continuation of equilibria to ODEs, but also the continuation of periodic solutions of ODEs, fixed points of discrete dynamical systems, and even solutions to partial differential equations (PDEs) that can in some sense be transformed to ODEs, like spatially uniform solutions (i.e. solutions that do not depend on any spatial variable) of a system of parabolic[9] partial differential equations (parabolic PDEs), travelling wave solutions to a system of parabolic PDEs, and even more.

It is presently not of our interest *how* AUTO finds this solution. For convenience, we only note here that all continuation methods basically rely upon some version of Newton's method (and therefore the Implicit Function Theorem).

We want to stress that continuation always leads to a (discretized) *continuum* of solutions. This is an advantage with respect to the other numerical methods we described so far. Moreover, a continuation and bifurcation package such as AUTO is able to detect bifurcations of the system as well. This is the second main advantage.

We show the method of continuation applied to our equilibrium problem which consists of a nonlinear ordinary differential equation which is difficult to solve analytically. The nonlinear differential equation for which the voltage V and capacitance C are calculated, reads

$$\frac{\partial^4 u}{\partial x^4} = -\frac{V^2}{2}\frac{\epsilon_0}{(u+d/\epsilon)^2} + \alpha k_1 \mathrm{e}^{-k_2 u} \tag{4.16}$$

with $u'(0) = u'(1) = u(0) = u(1) = 0$ and α a dummy parameter to switch between nonlinear $\alpha = 1$ and linear problem $\alpha = 0$. Setting $\alpha = 0$, the associated linear problem is obtained as

$$\frac{\partial^4 u}{\partial x^4} = -\frac{V^2}{2}\frac{\epsilon_0 \epsilon}{d} \tag{4.17}$$

with $u''(0) = u''(1) = u(0) = u(1) = 0$.

[9]We do not explain the notion of a *parabolic* PDE here; it is of no importance to us. But see any introductory text on partial differential equations

By solving the above linear equation, AUTO knows a solution of the "nonlinear" problem for $\alpha = 0$. By continuation in α, it subsequently finds solutions for the nonlinear problem with $\alpha \neq 0$. For each of these solutions the capacitance C and voltage V are calculated and a CV-curve is plotted in Figure 4.5. The CV-curve in Figure 4.5 exhibits a hysteresis behaviour.

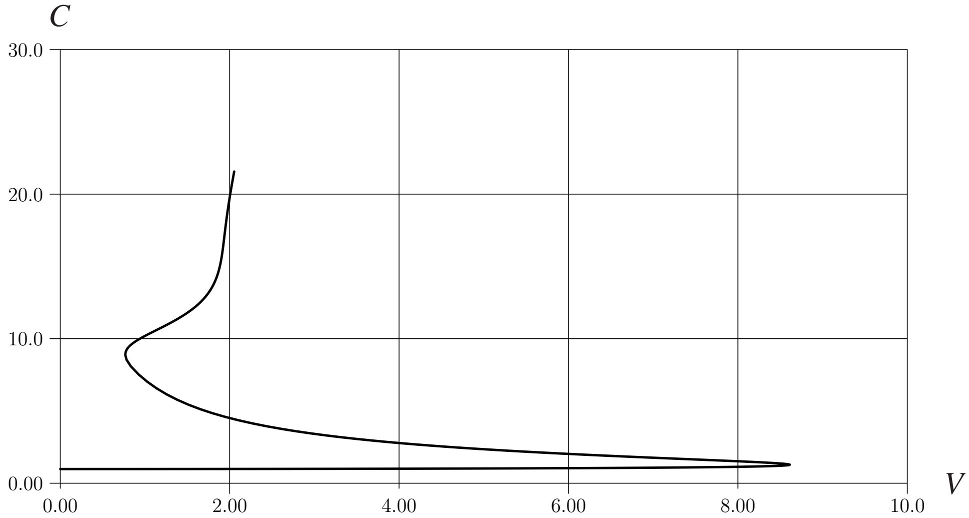


Figure 4.5: CV-curve generated by AUTO.

4.4 Analytical results

4.4.1 The linearized problem

It is possible to fully solve the linearized problem for three different cases: (i) the case in which the membrane does not touch the dielectric at all (ii) the case in which the membrane touches the dielectric in one point only and (iii) the case in which the membrane touches the dielectric on an interval. Since most linear problems have unique solutions, it is clear from the outset that the typical hysteresis behaviour does not show up in the linearized model. Some of those calculations may nevertheless be of interest, we have placed a summary of the linearized problem in the appendix

4.4.2 Collected analytical results

We prove some results for a functional E that may be interpreted as the total energy. The functional can be written as an integral over some domain Ω in $\mathbb{R}^2$. To read this section, it might be necessary to consult a text on variational methods, see for example [4] or [5].

First, it is proved that the solution for the membrane cannot touch the dielectric "with holes", i.e. in one dimension, the membrane is stuck to the dielectric between every two points where the membrane touches it. Second, it is derived that every critical point u for which $u = 0$ on some open set $\Omega_1 \subset \Omega$, has $\Delta u = 0$ on $\partial\Omega_1$. Third, we prove that stationary solutions for the energy E for which it holds that $dC/dV < 0$ are necessarily unstable. The final result is argued but still not completely proved. It states that if for both large V and small V a unique critical point exists, then under some conditions on the energy functional, a continuous family of solutions connects the two solutions.

Just for notation's sake: the main functional we consider is

$$E = \frac{D}{2}\int_\Omega u''^2 - \frac{V^2}{2}\int_\Omega \frac{\epsilon_0}{(u + d/\epsilon)} + \int_\Omega k_1 \mathrm{e}^{-k_2 u}, \tag{4.18}$$

where Ω is a domain (e.g. a rectangle, or a circle) in $\mathbb{R}^2$ or an interval in $\mathbb{R}$, depending on the question considered. The second integral is the *capacity*

$$C = \int_\Omega \frac{\epsilon_0}{(u + d/\epsilon)}.$$

The boundary conditions are $u = g$ and $\partial u/\partial n = 0$ on $\partial\Omega$. Unless stated otherwise, all integrals are over Ω.

Short list of results

1. For any minimizer (or general critical point) u of the *infinitely-hard bottom* problem

$$\min\left\{\frac{D}{2}\int \Delta^2 u - \frac{V^2}{2}C \;\middle|\; u \geq 0\right\} \tag{4.19}$$

   there exists *no* nonempty open sets $\Omega_1 \subset \Omega$ satisfying $u|_{\Omega_1} > 0$ and $u|_{\partial\Omega_1} = 0$. In particular, in dimension $n = 1$, the contact set $\{x \in \Omega \,|\, u(x) = 0\}$ is a (possibly empty) interval; in two-dimensions it means that the contact set has only simply connected components (no rings).

2. If u is minimizer of (4.19), or more generally a critical point, then if $u = 0$ on an open set $\Omega_1 \subset \Omega$, then $\Delta u = 0$ on $\partial\Omega_1$ (also of course on the interior of Ω_1).

3. Stationary points of E lying on a branch for which $dC/dV < 0$, are necessarily *unstable*. That is, there exists a perturbation w such that

$$E''(u) \cdot w \cdot w < 0.$$

   More generally, consider energies of the form

$$F(u, C, V) = \int f(x, u, \nabla u, \Delta u, \ldots)dx + G(V, C),$$

where $C = \int c(u(x))dx$ and V is a parameter (the Voltage for example). Then if $\partial^2 G/\partial C \partial V < 0$, any solution lying on a branch for which $dC/dV < 0$, is unstable.

4. The last result is more tentative; it should be true, but requires additional work to prove: if there exists a unique critical point of E both for small V and for large V, and provided that some type of coercivity holds for the energy functional (4.18), then there exists a continuous family of solutions connecting these two.

Sketches of the proofs

Ad 1 In 1D: let u be a stationary point, satisfying, where $u > 0$,

$$-Du'''' = \frac{\epsilon_0}{(u + d/\epsilon)^2}. \tag{4.20}$$

Note that the right hand side of (4.20) is strictly positive. Suppose u has two contact points $x_1 < x_2$. Since $u(x_i) = u'(x_i) = 0$ and $u \geq 0$, we must have $u''(x_i) \geq 0$. Furthermore, $-(u'')'' > 0$ and it follows from the maximum principle that $u''(x) \geq \min\{u''(x_1), u''(x_2)\} \geq 0$ for $x \in (x_1, x_2)$. This implies, again by the maximum principle, that $u \leq 0$ on (x_1, x_2). We thus conclude that $u \equiv 0$ on $[x_1, x_2]$.

In more dimensions exactly the same (pair of maximum principle) arguments prove that the contact region can have no holes, as asserted.

Ad 2 We do not give a full proof, but illustrate the main idea. On a one-dimensional domain $\Omega = [-L, L]$, let $u_R(x)$ be a smooth family of symmetric solutions with "forced" contact region $[-R, R]$, with $R < L$. By symmetry, we only need to consider the left half of the solution:

$$\begin{cases} -Du_R'''' = \frac{\epsilon_0}{(u_R + d/\epsilon)^2} & \text{for } -L < x < -R, \\ u_R(-L) = g,\ u_R'(-L) = 0, \\ u_R(-R) = 0,\ u_R'(-R) = 0. \end{cases}$$

Now, u_R is a critical point of E if and only if $dE(u_R)/dR = 0$.

Writing $E(u) = \int_\Omega \frac{D}{2}u''^2 + g(u)dx$, where $g(u) = -\frac{V^2}{2}\frac{\epsilon_0}{(u+d/\epsilon)} + k_1 \mathrm{e}^{-k_2 u}$, we obtain

$$E(u_R) = 2\int_{-L}^{-R} \frac{D}{2}u_R''^2 dx + 2\int_{-L}^{-R} g(u_R)dx + 2Rg(0).$$

Calculating this derivative with respect to R we infer that

$$\frac{dE(u_R)}{dR} = E_u(u_R)\frac{\partial u_R}{\partial R} - Du_R''^2(-R) - 2g(u_R(-R)) + 2g(0) = -Du_R''^2(-R),$$

since $u_R(-R) = 0$ and $E_u(u_R) = 0$, because u_R is a critical point when keeping R *fixed*. It follows that $u''(-R) = 0$ if u is a critical point of E.

This argument can be extended to higher dimensions quite easily, under the *assumption* that the solution for fixed contact region varies smoothly with the geometry of the contact region. Without this assumption, more complicated arguments are needed.

Remark 4.4.1. We note that if the contact set is a single point, then the second derivative in this point need *not* be zero. Indeed, in one dimension for example, there is a branch of solutions with contact only in the midpoint of the domain (interval) and with varying second derivative.

Ad 3 Let us first give the argument for the specific energy E in the one-dimensional case. Consider

$$\frac{D}{2}\int u''^2 - \frac{V^2}{2}\int \frac{\epsilon_0}{(u+d/\epsilon)} + k_1 \mathrm{e}^{-k_2 u}.$$

Let us look at stationary points, i.e., solutions of

$$Du'''' = -\frac{V^2}{2}\frac{\epsilon_0}{(u+d/\epsilon)^2} + k_1 k_2 \mathrm{e}^{-k_2 u}, \tag{4.21}$$

which are, on the branch under consideration, parametrized by V. Let us write $\overline{u}$ for the derivative of the solutions u with respect to V along the branch. Taking the derivative of (4.21) along the branch, we obtain

$$D\overline{u}'''' = V^2 \frac{\epsilon_0 \overline{u}}{(u+d/\epsilon)^3} - V\frac{\epsilon_0}{(u+d/\epsilon)^2} - k_1 k_2^2 \mathrm{e}^{-k_2 u}\overline{u}. \tag{4.22}$$

The second variation of the energy in the direction $\overline{u}$ gives

$$E''(u)\cdot\overline{u}\cdot\overline{u} = D\int \overline{u}''^2 - V^2\int \frac{\epsilon_0 \overline{u}}{(u+d/\epsilon)^3} + k_1 k_2^2 \int \mathrm{e}^{-k_2 u}\overline{u}^2.$$

After performing partial integration twice on the first term, we can substitute (4.22) and, with most terms cancelling, we obtain

$$E''(u)\cdot\overline{u}\cdot\overline{u} = -V\int \frac{\epsilon_0 \overline{u}}{(u+d/\epsilon)^2}.$$

This simplifies as

$$E''(u)\cdot\overline{u}\cdot\overline{u} = -V\int \frac{\epsilon_0 \overline{u}}{(u+d/\epsilon)^2} = VC'(u)\overline{u} = V\frac{dC}{dV}.$$

Hence $\frac{dC}{dV} < 0$ implies that u is unstable.

For the general case, critical points $u = u(V)$ satisfy, subscripts denoting partial derivatives,

$$F_u(u(V))\cdot w + G_C(V, C(u(V)))\, C_u(u(V))\cdot w = 0 \qquad \text{for any } w.$$

Taking the derivative with respect to V gives (always evaluating at $u = u(V)$),

$$F_{uu} \cdot w \cdot u_V + G_{CV}\, C_u \cdot w + G_{CC}\, C_u \cdot w\, C_u \cdot u_V + G_C\, C_{uu} \cdot w \cdot u_V = 0 \quad \text{for any } w. \tag{4.23}$$

On the other hand, the second variation (for fixed V) gives

$$F''(u) \cdot w \cdot w' = F_{uu} \cdot w \cdot w' + G_{CC}\, C_u \cdot w\, C_u \cdot w' + G_C\, C_{uu} \cdot w \cdot w',$$

hence, using (4.23), we obtain for $w = w' = u_V$

$$F''(u) \cdot u_V \cdot u_V = -G_{CV}\, C_u \cdot u_V.$$

When we write $\overline{C}(V) = C(u(V))$, this reduces to

$$F''(u) \cdot u_V \cdot u_V = -G_{CV} \frac{d\overline{C}}{dV}.$$

Hence, if $\partial^2 G/\partial C \partial V < 0$ then solutions on branches where $d\overline{C}/dV < 0$ are always unstable.

Remark 4.4.2. One can also consider the problem where we put a charge $Q = VC$ on the switch. In that case the physically relevant energy does *not* include the energy stored in the battery, which is given by $-V^2 C$. The energy E_Q thus becomes

$$E_Q = \frac{D}{2} \int u''^2 + \frac{Q^2}{2C} + \int k_1 \mathrm{e}^{-k_2 u},$$

and the arguments above show that solutions on curves with $dC/dQ < 0$ are always unstable.

Ad 4 Such a result follows from degree theory, see e.g. [8]. However, it still needs to be checked rigorously that there indeed does exist a unique critical point for very large and very small V. For $V = 0$ this is obvious, the energy being convex in that case, but the situation for large V is less straightforward, since the energy contains both convex and concave parts, although in numerical experiments uniqueness is observed.

4.4.3 Functional estimates

In this section, two estimates for the first and second variation of the total energy are derived.

The energy functional modeling the deformation of a clamped plate Ω under influence of an electrical field due to a potential difference with a fixed plate reads

$$\begin{aligned} E[u] &= E_{\text{mech}} + E_{\text{el}} \\ &= \int_\Omega \left[\frac{1}{2} D(\Delta u)^2 - \frac{1}{2} V^2 \frac{\varepsilon_0}{u + g + \frac{d}{\varepsilon_0}} \right] \mathrm{d}x\mathrm{d}y, \end{aligned} \tag{4.24}$$

where $u \in H_0^2(\Omega)$, implying $u = \frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. Equilibria for the system are the zeroes of the first variation,

$$\delta E[\tilde{u}, h] = 0, \quad \forall h \in H_0^2(\Omega).$$

These can be stable and unstable equilibria (e.g. saddle points). A stable equilibrium is a minimum of the functional. Such states are characterized by the fact that the second variation at the equilibrium state is strictly positive,

$$\delta^2 E[\tilde{u}, h] > 0, \quad \forall h \in H_0^2(\Omega)$$

We here wish to give a sufficient condition for an equilibrium to be stable.

The first variation is found by putting $u = \tilde{u} + \varepsilon h$, where $h \in H_0^2(\Omega)$ is a test function, and taking the derivative with respect to. ε at $\varepsilon = 0$. We then obtain

$$\delta E[\tilde{u}, h] = \int_\Omega \left[D\Delta^2 \tilde{u} + \frac{1}{2} V^2 \frac{\varepsilon_0}{(u + g + \frac{d}{\varepsilon_0})^2} \right] h \, \mathrm{d}x\mathrm{d}y.$$

The variation lemma yields the boundary value problem for the system from this functional. Let us assume we have a solution for the system. Now the question is whether the solution is stable or not. The second variation in the direction $h \in H_0^2(\Omega)$ is found to be

$$\delta^2 E[u, h] = \int_\Omega \left[D(\Delta h)^2 - V^2 \frac{h^2 \varepsilon_0}{(u + g + \frac{d}{\varepsilon_0})^3} \right] \mathrm{d}x\mathrm{d}y. \tag{4.25}$$

In this form it is difficult to check positivity. However, we can prove a Cauchy-type inequality for the test functions in the space $H_0^2(\Omega)$ when Ω has a simple shape. For the case of a rectangle with sides L_1 and L_2 we have

$$\int_\Omega (\Delta h)^2 \mathrm{d}x\mathrm{d}y \geq \frac{4}{\max[L_1^4, 2L_1^2 L_2^2, L_2^4]} \int_\Omega h^2 \mathrm{d}x\mathrm{d}y$$

Using this inequality together with equation (4.25) we have the following estimate for the second variation,

$$\delta^2 E[u, h] \geq \int_\Omega \left[\frac{4D}{\max[L_1^4, 2L_1^2 L_2^2, L_2^4]} - V^2 \frac{\varepsilon_0}{(u + g + \frac{d}{\varepsilon_0})^3} \right] h^2 \mathrm{d}x\mathrm{d}y$$

Necessary conditions for the stability of the functional can be obtained from this estimate. For example, take $u^* = \min(u)$, then

$$\delta^2 E[u, h] \geq \left[\frac{4D}{\max[L_1^4, 2L_1^2 L_2^2, L_2^4]} - V^2 \frac{\varepsilon_0}{(u^* + g + \frac{d}{\varepsilon_0})^3} \right] \int_\Omega h^2 \mathrm{d}x\mathrm{d}y,$$

and it is sufficient to check the positivity of the constant.

Appendix

As said in section 4.4.1, the linearized problem can be fully elaborated for three different cases: (i) the case in which the membrane does not touch the dielectric at all (ii) the case in which the membrane touches the dielectric at $x = 0$ only and (iii) the case in which the membrane touches the dielectric on an disc $\bar{B}_a$, for $0 < a < 1$. We will make a few remarks on how to do this, in the case of a radially symmetric MEMS switch. We consider case (iii). It will be clear from the result that the typical hysteresis behaviour does not show up in the linearized model.

To focus on the right parameter combinations in the problem, we rescale it. For example, in the 2-D radially symmetric version of the problem one obtains for the capacitance:

$$C(w) = \frac{2\pi\epsilon_0\Lambda^2}{g}\int_0^1 \frac{r}{1+\eta+w(r)}\,dr.$$

and, by computing the Euler-Lagrange equation corresponding to this energy we find

$$\Delta_r^2 w = -\frac{\delta v^2}{(1+\eta+w)^2}. \tag{4.26}$$

where $\Delta_r = \frac{1}{r}\frac{d}{dr}$, δ some algebraic expression in terms of the other parameters, v a non-dimensionalized voltage and w a scaled version of the distance u. This problem can subsequently be linearized around $w = 0$:

$$\Delta^2 w = \omega^4\left(-\frac{1+\eta}{2} + w\right),$$

where $\omega = \sqrt[4]{\frac{2\epsilon v^2}{(1+\eta)^3}}$ is just a scaling. Regarding w as a radially symmetric function depending on r only we get

$$w(r) = AJ_0(\omega r) + BY_0(\omega r) + CI_0(\omega r) + DK_0(\omega r) + \frac{1+\eta}{2}, \tag{4.27}$$

where J_0 and Y_0 are Bessel functions of the first and second kind respectively and I_0 and K_0 are modified Bessel functions of the first and second kind. We add the following boundary conditions:

$$w(1) = w'(1) = w'(a) = w''(a) = 0, \quad w(a) = -1.$$

By rewriting this system as a four-dimensional first-order system, one obtains the constants A, B, C and D.

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