# Multiple delay differential systems in a sensing problem


**Problem presented by**

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**Problem statement**

An active sensing system probes the environment by transmitting a signal and processing the output of a receiver, which generally contains echoes from reflectors and boundaries, and also noise. Usually the transmission is pulsed. Delay between the transmission of a pulse and the moment when it returns to the receiver determines the distance to a target. Using a continuous periodic signal in transmission leads immediately to ambiguities and demasking. The scheme under consideration here transmits a continuous signal generated by a driven chaotic dynamical system, in an effort to reduce ambiguity and to increase robustness against noise.


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# 1 Statement of the problem

An active sensing system probes the environment by transmitting a signal and processing the output of a receiver, which contains echoes from reflectors and boundaries. It will also contain noise. Usually the transmission is pulsed. Delay between the transmission of a pulse and the moment when it returns to the receiver determines the distance to a target. Using a continuous periodic signal in transmission leads immediately to ambiguities and demasking. The scheme under consideration here transmits a continuous signal generated by a driven chaotic dynamical system. There are the following advantages in the use of chaotic signals:

**(1) Reduction of ambiguity:** Usually one estimates the time delay $\tau$ between pulsed or continuous signals by shifting them against each other and comparing. For some $\tau$ the two signals are most similar, as assessed by means of statistical characteristics like the mutual correlation function (dynamical methods can be used as well). However, two periodic signals will be most similar for $\tau + kT$, where $k$ is an integer and $T$ is the period of the signal. Thus, there will always be $kT$ uncertainty when defining the time delay $\tau$ from periodic signals. On the contrary, chaotic signals do not repeat in time and thus avoid this ambiguity.

**(2) Robustness:** A chaotic signal possesses a broad-band Fourier power spectrum. Information theory testifies that broad-band signals are more robust against external noise [1].

**(3) Security:** It is always more difficult to recognize a chaotic signal than a periodic one.

A means of determining echo arrivals is required. Finding the correlation function and use of higher order statistics are the methods put forward by Bauer [2, 3]. In this report, a scheme is proposed, based on the work of Thompson [4], by which the presence of a signal causes bifurcation in a dynamical system. There are two versions. In the first version, the delayed and attenuated output of the transmitter is fed back into itself (Fig. 1(a)). In the second version, two dynamical systems run in parallel, with the delayed and attenuated output of the transmitter being fed into the receiver (Fig. 1(b)). Due to possible multipath propagation, the signal on the input of the receiver could be a mix of echoes with different intensities $\alpha_i$ and delays $\tau_i$.

As an idealisation we assume the medium in which signals propagate to be homogeneous, linear and non-dispersive. For the formulae below we also assume the bottom to be horizontal. A bottom with a slope or of a more complex shape will require more sophisticated geometrical formulae connecting the signal delay and attenuation with the configuration of the environment. However, once the configuration of the system is known, the corresponding formulae can be derived and substituted into the model equations. The basic feature of signal propagation will remain the same, independent of configuration: the longer the signal path is, the larger is the time shift between the radiated and received signals and the larger is the attenuation. Complication of the geometry will not affect considerations below.

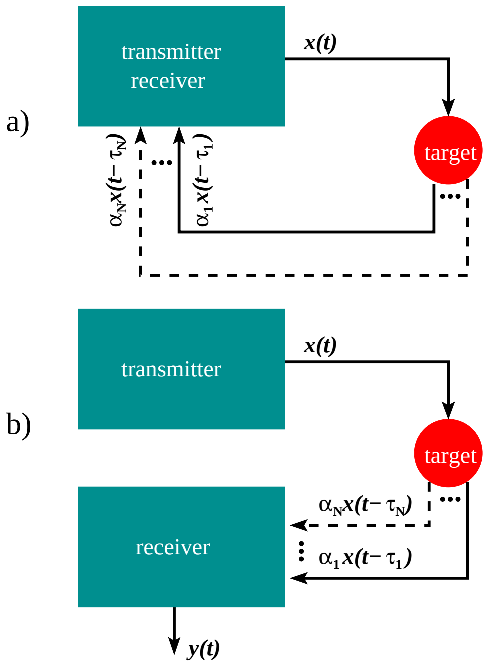


Figure 1: Two schemes for inducing bifurcations in a receiving system: (a) direct feedback scheme and (b) parallel systems.

Within this approximation, there can be two possibilities:

**(1)** The water only attenuates the propagated signals, and does not make other transformations. Let us take into account two reflected signals and introduce how their parameters will be related to the geometry of the whole system. One signal is reflected from the target and returns directly to the processing device (suppose this is roughly the same straight line along which the signal traveled to the target after being radiated). Then its attenuation factor $\alpha_0$ and time delay $\tau_0$ will be

$$\alpha_0 = 1/r, \;\; \tau_0 = r/c\,, \tag{1}$$

where $c$ is the wave speed and $r$ is the distance from the transmitter to the target, as shown in Fig. 2. The second signal will be reflected from the target at some nonzero angle and go to the water surface. It will then be reflected from the water surface, which is supposed to be perfectly reflecting. Its attenuation factor $\alpha_1$ and time delay $\tau_1$ will be

$$\alpha_1 = 2/\sqrt{r^2 + 4d^2}, \;\; \tau_1 = \sqrt{r^2 + 4d^2}/c, \tag{2}$$

with $d$ as shown in Fig. 2. If one takes into account the reflection from the bottom as well, the attenuation factor $\alpha_2$ and time delay $\tau_2$ will be

$$\alpha_2 = 2/r_2, \;\; \tau_2 = r_2/c\,, \tag{3}$$

where $r_2 = \sqrt{r^2 + 4(D-d)^2}$ and $D$ is as shown in Fig. 2.

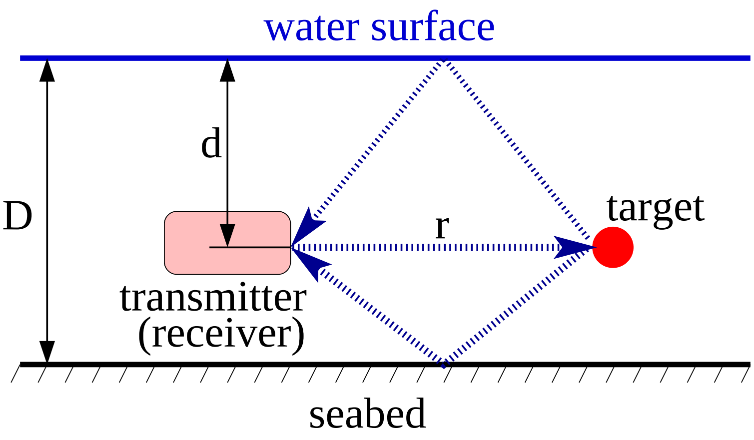


Figure 2: Geometry of two-path signal propagation.

**(2)** The water may make some linear transformation of the system resulting in a change to the Fourier spectrum. Note that linear transformations cannot remove or add spectral components. The only effect of linear transformations can be attenuation of some components to a larger extent than others. To characterize this transformation a transfer function can be used.

In this case we can estimate the transfer function of the medium and present a received signal not just as a sum of attenuated and delayed versions of the radiated one, but as a sum of convolutions of each reflected signal with the transfer function.

As a refinement to the above approximation, one can take into account nonlinear properties of the propagation medium, in agreement with *e.g.* [5]. The results of simulation with nonlinear media should be compared with the results for linear media to reveal the extent to which nonlinearity influences the operation of the whole system.

The problem is to analyse the behaviour of the proposed detection system and to establish conditions under which it bifurcates in the presence of a signal. Note that the model applies both to monostatic (source and receiver collocated) and bistatic systems, with suitable choice of the path parameters. Note also that this is a simplified case, because the source and target will not in general be at the same distance from the water surface. The key matter is the presence in the problem of the attenuations and delays. Their dependence on the environment may not be important for the initial investigation. For realization of scheme Fig. 1(a) a forced Duffing oscillator could be considered as an example of a transmitter. The model equations in the approximation of two-path propagation are then written as

$$\ddot{x} + c_1\dot{x} - c_2x + c_3x^3 = F\cos(\omega t) + \sum_{i=0}^{2}\alpha_i x(t-\tau_i) + \xi(t). \tag{4}$$

Here $c_{1,2,3}$, $F$ and $\omega$ are intrinsic system parameters; $\alpha_{0,1,2}$ and $\tau_{0,1,2}$ are the attenuations and delays with which the radiated signal is received, having been reflected from a target (see (2), (3)); $\xi$ is some generalized noise in the system.

The final aim of this technology is to find the distance from the target, and possibly other data like depth *etc.*, using knowledge about bifurcations in systems like (4). A special requirement is that this information should be gained from *short* data sets to speed up the process of parameter recognition.

# 2 Solution of the problem

The Study Group first defined the task more precisely from the nonlinear dynamics point of view. Then the requirements of the dynamical systems suitable for the presented problem were elaborated. Finally, possible ways of solution were proposed. All numerical results during discussion were obtained by using the "Dynamics Solver" software [6].

## 2.1 Problem definition

The whole problem can be split into three parts:

1. Signal radiation.
2. Signal propagation through water.
3. Processing of the reflected signal by a receiver.

In reality the signal is affected by noise at each of the above stages. Let us for simplicity consider the idealised noise-free case.

It was initially supposed that the radiated signal has to be chaotic. Therefore the dimension of possible models cannot be less than three. For simplicity, propagation of the signal through the water was considered as its linear transformation. Then a general model of the 'feedback scheme' of Fig. 1(a) could be written as

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, \ldots, x_N; t; \vec{\mu}) + \sum_i \alpha_i x_1(t - \tau_i) \\ &\cdots \\ \dot{x}_N &= f_N(x_1, x_2, \ldots, x_N; t; \vec{\mu}) \end{aligned} \tag{5}$$

and a general model of the 'parallel scheme' of Fig. 1(b) as

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, \ldots, x_N; t; \vec{\mu}) \\ &\cdots \\ \dot{x}_N &= f_N(x_1, x_2, \ldots, x_N; t; \vec{\mu}) \\ \dot{y}_1 &= g_1(y_1, y_2, \ldots, y_L; t; \vec{\nu}) + \sum_i \alpha_i x_1(t - \tau_i) \\ &\cdots \\ \dot{y}_L &= g_L(y_1, y_2, \ldots, y_L; t; \vec{\nu}) \end{aligned} \tag{6}$$

Here $\vec{x}$ is a state of radiating system, $\vec{y}$ is a state of receiving system, $\vec{\mu}$, $\vec{\nu}$ are vectors of system parameters, and $t$ is time. In terms of (5) and (6) the initial problem could be reformulated into the problem of finding $\alpha_i$ and $\tau_i$, if $\vec{x}$ ,$\vec{y}$, and $\sum_i \alpha_i x_1(t-\tau_i)$ are known. How in this situation could the knowledge about bifurcations in systems (5) and (6) be used?

The approach consists of plotting the (series of) bifurcation diagram(s) of the system on the plane of two selected control parameters. Bifurcation diagrams consist of the curves separating the regions with qualitatively different behaviour. Each type of behaviour is associated with an attractor. Assume a control parameter corresponding to the horizontal axis of the parameter plane is the distance from the target. Then one wants the bifurcation diagram to have a striped structure consisting of narrow adjoint regions, every two neighbouring regions corresponding to qualitatively distinct attractors. Then the received signal can be classified as belonging to one of these regions, and the distance from the target will be estimated with some finite accuracy defined by the width of this particular region in the parameter plane.

Therefore, the system modelling the transmitter should satisfy the following criteria:

**(1)** **Richness of the bifurcation diagram.** To provide a more precise estimation of the control parameter, one needs the different regions in the parameter plane to be narrow. This means that for a given range of control parameter values, the number of different regimes will be large enough. The requirement for short datasets to process also imposes restrictions to the system: its attractors should allow an easy and reliable detection from short datasets.

**(2)** **Robustness towards external fluctuations.** One needs the system to be robust towards noise. A problem is likely to arise from the effect of unavoidable noise on the system, especially near the bifurcation. When the system's parameters are selected near the bifurcation curve, but bifurcation has not yet occurred, noise can induce the bifurcation. This means an effective shift of parameter value and will lead to incorrect estimation of the corresponding control parameter. In view of this, regions in the parameter plane should be not too narrow, to avoid frequent closeness of the system to their borderlines. This requirement contradicts the first one. Thus, a balance between these two requirements is needed.

**(3)** **Absence of multistability.** Multistability is the coexistence in the phase space of different attractors for the same set of control parameters. In practice this means that different initial conditions may lead to distinct attractors. Moreover, for different attractors there are different bifurcation curves in the parameter plane. Also noise can induce random switches between the two attractors so that the resulting motion will not be simply smeared motion on one attractor, but one which would never exist without noise. Multistability will thus cause strong ambiguity in defining the control parameter value.

## 2.2 Choice of the models for transmitter and receiver

From the practical point of view, the system under consideration should be as simple as possible. The minimal dimension of dynamical system being able to generate chaotic oscillations is three, and therefore we started our considerations from three-dimensional models of chaotic systems. Then, for realization of the 'feedback scheme' a three-dimensional system (two-dimensional nonautonomous) is sufficient to generate chaotic signals, whereas for the 'parallel scheme' the dimension of the system should *a priori* be more than three, because of additional equations required for the description of the receiving system. Therefore, consideration was started from the feedback scheme.

### 2.2.1 Feedback scheme

During discussion, two systems were studied, namely the two-dimensional Duffing oscillator subject to harmonic forcing and three-dimensional autonomous Rössler oscillator. The corresponding equations read as follows:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -c_1 y + c_2 x(1-x^2) + F\cos(t) + \alpha_0 x(t-\tau_0) + \alpha_1 x(t-\tau_1) + \alpha_2 x(t-\tau_2) \end{aligned} \tag{7}$$

for Duffing oscillator and

$$\begin{aligned} \dot{x} &= -y - z + \alpha_1 x(t-\tau_1) + \alpha_2 x(t-\tau_2) \\ \dot{y} &= x + ay \\ \dot{z} &= a + z(x-b) \end{aligned} \tag{8}$$

for the Rössler system. The parameters $\alpha_{0,1,2}$ and $\tau_{0,1,2}$ reflect the geometry of the transmitter-target system in Fig. 2 and are defined by formulae (2) and (3). Bifurcations in (7) and (8) have been studied with respect to variation of the parameter $r$. It was found that in both cases there are wide ranges of parameters where the models under consideration demonstrate rich bifurcation structures formed by transitions between different types of periodic, quasiperiodic and chaotic motions. In Fig. 3 examples of bifurcation diagrams for both models are shown. The value of parameter $r$ is shown on the horizontal axis, and the Poincaré section of system's solution is presented along vertical axis.

Thus, from the viewpoint of richness of bifurcations both system are suitable for solution of the given problem. However, our studies have also shown that the Duffing system is able to demonstrate a multistability in a wide range of parameter values. Multistability manifests itself in the coexistence of different types of attractors for the same parameter values and makes a solution depend on initial conditions. The latter leads to ambiguities. In Fig. 4, three coexisting attractors are depicted for the same parameter values. Actually, the occurrence of multistability in the Duffing model is in many respects a result of the specific nonlinearity of Duffing system, due to which the Duffing system possesses three equilibrium states. Around two of them the rotation of phase trajectories is possible as well as possible rotation around all the three equilibrium points. In this respect, the Rössler system seems to be preferable for the specified aims, since only two equilibrium points exist for that system, and the attractor corresponds to rotations only around one of them.

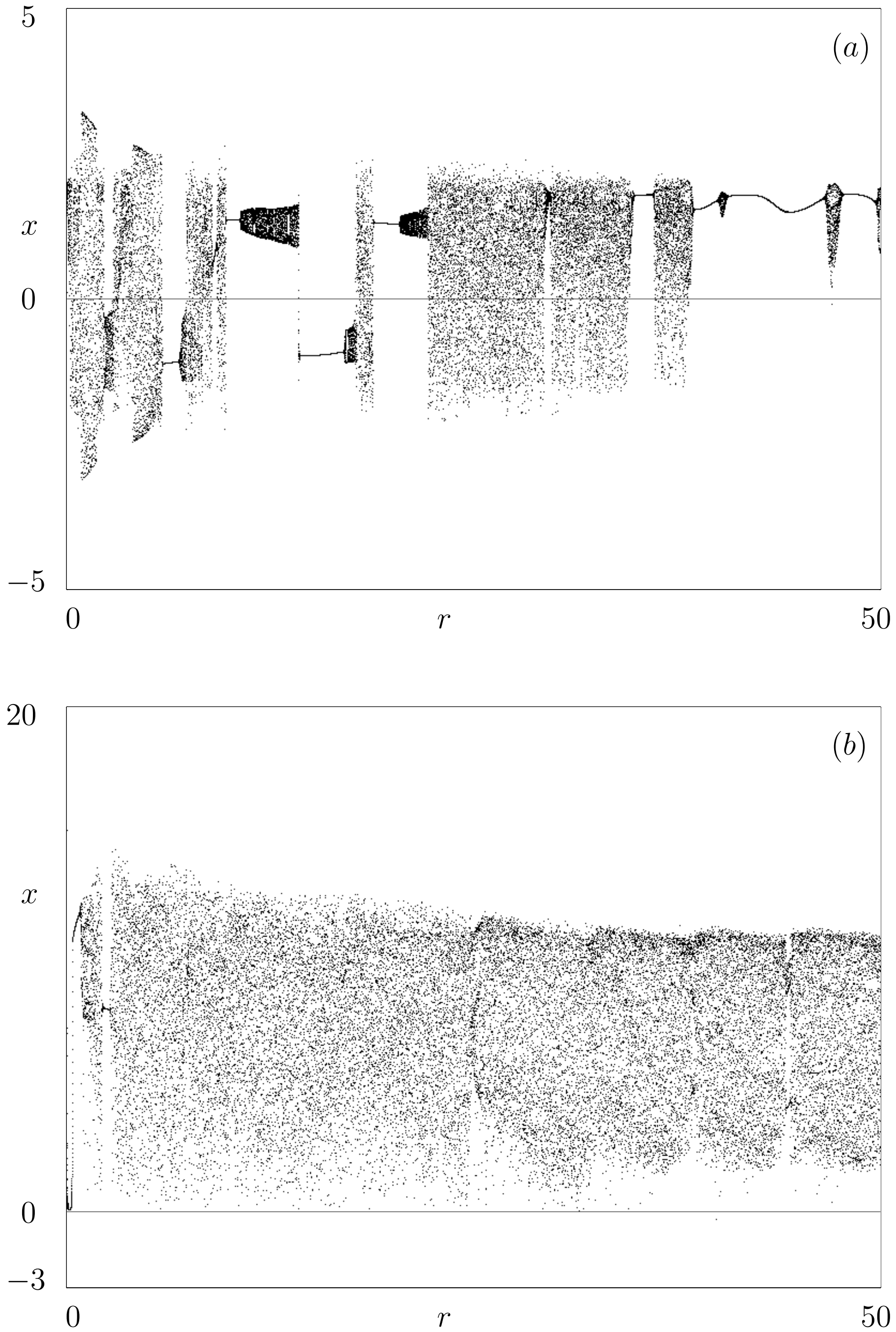


Figure 3: Examples of bifurcation diagrams for (a) a forced Duffing oscillator with parameters $c_1 = 0.04$, $c_2 = 1$, $F = 0.3$ using a stroboscopic Poincaré section with $t = 2\pi n + \text{const}$ and (b) the Rössler system with parameters $a = 0.2$, $b = 6.5$ using a Poincaré section with $y(t) = 0$. Parameters $\alpha_{0,1,2}$ and $\tau_{0,1,2}$ were defined for $c = 1$, $D = 10$, and $d = 8$.

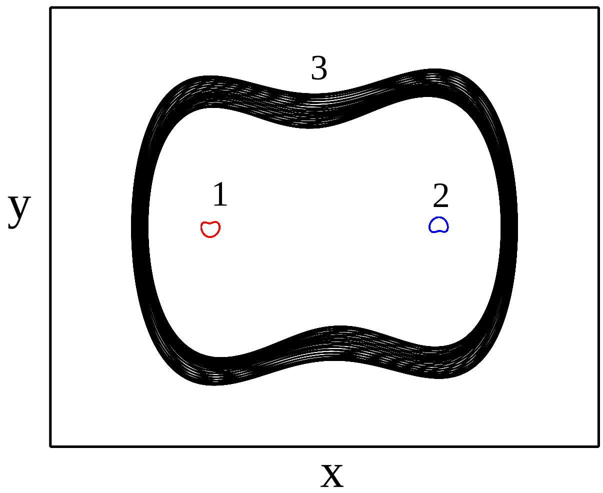


Figure 4: Multistability in a forced Duffing oscillator with delayed feedback for parameters values $c_1 = 0.24$, $c_2 = 1$, $F = 0.27$, $r = 2$, $D = 10$, $d = 8$.

**Disadvantages of the feedback scheme:**

(1) The experience of plotting bifurcation diagrams for various systems shows that one of the typical delay-induced regimes is expected to be a limit cycle whose shape changes from region to region. A limit cycle corresponds to a periodic behavior of the system. This means that since the radiated signal is defined by the existing attractor within this scheme, it will also be periodic. This will result at least in violation of masking for radiated signals.

(2) The knowledge of the bifurcation diagram will allow one to define the distance from the target only within the width of the particular region in the plane of control parameters. Thus, it will not be a precise definition.

### 2.2.2 Parallel scheme

To overcome the first disadvantage of the previous approach, the parallel scheme was considered. The first system only generates and emits a chaotic signal, while the reflected signal is fed to another system. This approach allows one to control precisely the characteristics of the signal propagating in water. With this, one can define the range of control parameters with the help of the bifurcation diagram in essentially the same way as in the previous case. As a receiver, one can select a system that satisfies the following requirements:

(1) Its bifurcation diagram is striped.

(2) Its bifurcation diagram is rich in attractors.

(3) Its attractors should allow reliable recognition from short datasets.

(4) There is no multistability.

From the general point of view, this scheme corresponds to the problem of two unidirectionally coupled nonlinear systems, where at least one of them (transmitter) is a self-sustained dynamical system with chaotic oscillations. In this kind of a scheme it is impossible to expect bifurcational transition from chaos to some regular one. Only transitions from one type of chaos to another are possible. It is generally impossible to distinguish reliably between different types of chaos using only short datasets, therefore condition 3 above is violated. A possible way to surmount this complication is to try to use some phenomena which are induced in coupled self-sustained systems.

**Anticipating synchronization**

We propose to exploit a phenomenon which was recently discovered [7, 8] in delay differential equations and called *anticipating synchronization*. This phenomenon consists of the following. Assume we have two identical dynamical systems with chaotic attractors. The requirement for identity is essential here. Suppose a signal from one system is time delayed with delay $\tau$ and applied to another system. Note that this is a unidirectional coupling between the two identical systems. If the strength of the coupling is large enough, one observes that the state of the forced system $\vec{x} = x_1, x_2, \ldots, x_N$ at time $t$ is equal precisely to the state of the forcing system $\vec{y} = y_1, y_2, \ldots, y_N$ at time $t-\tau$, and this is valid for all times when the system's behavior is stationary. This phenomenon is called anticipating synchronization and can be very easily detected by observing the space (plane) of coordinates $\{x_1(t)-y_1(t-\tau),\ x_2(t)-y_2(t-\tau),\ \ldots,\ x_N(t)-y_N(t-\tau)\}$. If there is no synchronization, the phase trajectory wanders chaotically. If synchronization takes place, one observes just a fixed point in this space (plane). There seem to be no other types of bifurcations of chaotic attractors detected in the system studied in [7], and the only bifurcation transition is abrupt. Two unidirectionally coupled Rössler systems have been examined:

$$
\begin{aligned}
\dot{x_1} &= -y_1 - z_1 \\
\dot{y_1} &= x_1 + ay_1 \\
\dot{z_1} &= a + z_1(x_1 - b) \\
\dot{x_2} &= -y_2 - z_2 + \eta \\
\dot{y_2} &= x_2 + ay_2 \\
\dot{z_2} &= a + z_2(x_2 - b).
\end{aligned}
\tag{9}
$$

Here $a = 0.2$ and $b = 6.5$. The coupling term with a single delay of the form

$$
\eta = h(x_1(t-\tau) - x_2)) \tag{10}
$$

was used during the Study Group discussion. In terms of the initial problem, the latter means that one-path propagation of the signal takes place. It was shown numerically that the anticipating synchronization appears for coupling value $0.2 < h < 4$ for $r = 10$.

The possible way to use this phenomena for solution of the initial problem was elaborated in discussions. Let us require the coupling term $\eta$ to be in the form $\eta = h(a\alpha x(t-\tau) - x(t))$, where the values of $h$ and $a$ could be controlled, while $\alpha$

and $\tau$ correspond to the attenuation and time delay of signal which was received by system. The parameter $h$ is chosen to have such a value which guarantees the occurrence of anticipating synchronization in the case $a\alpha = 1$ for a large enough range of delays. We assume that attenuation of the reflected signal is tightly linked to the value of the time delay through the distance to the target. Assume that a signal with delay $\tau_0$ and attenuation $\alpha_0$ is fed into the receiving system. Using phase projection techniques one can detect if synchronization occurs or not. Suppose there was no synchronization originally. Let a factor (which may implemented automatically) multiply this signal by some $a > 1$. At a value of $a = 1/\alpha_0$ synchronization occurs. Then the value of attenuation $\alpha_0$ provides one with the length of a path of signal propagation.

Following the Study Group a generalised coupling term was considered with three delays of the form

$$\eta = h(\phi_1 - \phi_2), \tag{11}$$

where

$$\begin{aligned} \phi_1 &= \alpha_0 x_1(t-\tau_0) + \alpha_1 x_1(t-\tau_1) + \alpha_2 x_1(t-\tau_2) && (12)\\ \phi_2 &= \alpha_0 x_2 + \alpha_1 x_2(t-(\tau_1-\tau_0)) + \alpha_2 x_2(t-(\tau_2-\tau_0)), && (13) \end{aligned}$$

and remarkably the synchronization was also observed. Using (11) the differences between asynchronous and synchronous dynamics of (9) are illustrated by Fig. 5, where different phase projections of chaotic attractors are shown.

**Issues for further investigation**

(1) Since this phenomenon was just discovered for a single delay, little is known about its generality and robustness to noise. Subsequent work based on (11) seems to indicate that anticipating synchronization can occur in a system with multiple delays. However further research is needed to confirm this observation.

(2) If the basic assumption that the attenuation factor depends strongly on the delay is not true, that is, if this relation is very weak, the precision of this approach decreases.

## 2.3 Alternative methods of problem solution

As was mentioned above, all approaches based on bifurcation phenomena have these or other disadvantages. Therefore, the consideration of alternative (complementing) methods of solving the given problem seem to be very useful.

### 2.3.1 Nonlinear time series analysis

There is the possibility of using standard nonlinear time series analysis. One can feed the received chaotic signal into the receiver, register the output of it and compute from this output several dynamical or statistical characteristics like attractor dimension, Lyapunov exponents, rotation numbers, *etc.* One should do this for the whole range of possible $\alpha_i$ and $\tau_i$ for which bifurcation diagrams were planned to be plotted. One needs to

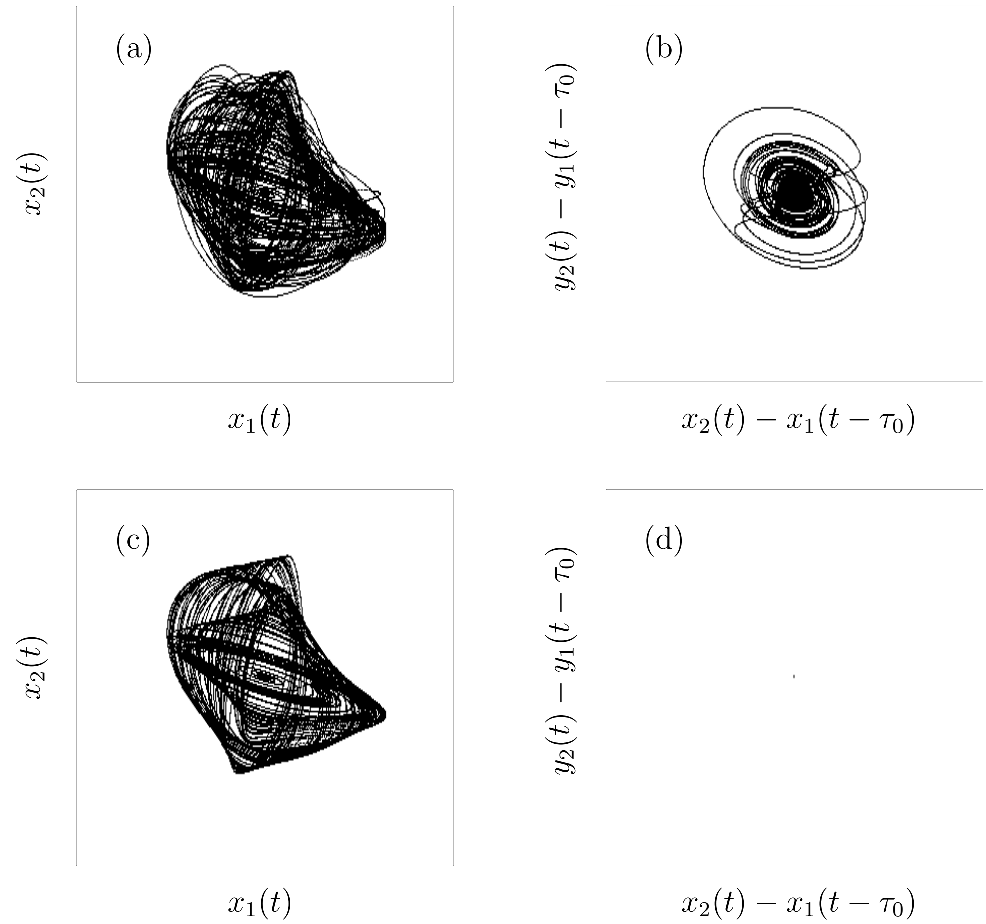


Figure 5: Phase projections of (a),(b) 'asynchronous' ($h = 1.0$, $r = 10$) and (c),(d) 'synchronous' ($h = 1.5$, $r = 10$) chaotic attractors of system (9).

associate with each point in the parameter plane the particular set of values of these characteristics. Finally, when a real experimental signal comes, one can simply compare this set estimated from the output of the receiver with the pre-calculated sets. If all characteristics coincide with the ones for a particular set of control parameters $\alpha_i$ and $\tau_i$, the latter presents the solution sought.

**Disadvantages:**

- This method requires a long dataset to process and not a negligible time of computation. This violates the requirement about short datasets and low processing time.
- The precision of estimation of any dynamical characteristics from noisy experimental signals is quite low.

### 2.3.2 Global reconstruction of equations of motion

Calculations of statistical and dynamical characteristics of available signals can overcome disadvantages of the bifurcation approaches, but the solution provided still does not satisfy some essential requirements for the data processing. In particular, it suggests treating the received signals as those coming from some black box and does not take into account that we know quite a lot both about the transmitter and about the receiver. Here we suggest a method to exploit this information by involving an approach referred to as global reconstruction of equations of motion from experimental data.

The task of global reconstruction is posed in its most general form as follows [9]. Assume we have a black box, about which nothing is known. At the output there is one or several signals which we can record and process. We assume that

- Output signals demonstrate sustained oscillations in time.
- The black box represents some finite-dimensional dynamical system governed by evolution equations, possibly with noise.
- The black box is a self-sustained system possessing an attractor. The presence of attractor is essential here in order to assume that our oscillations are always bounded.

The task in its original form is posed as follows: to write the equations of motion that are able to reproduce the signals at the output. This is called the problem of global reconstruction of equations of motion. However, if underlying attractors are chaotic, this is often impossible for a series of reasons whose discussion can be found in a variety of related publications [10]. Thus, less rigorous requirements for the reconstructed system are:

- The system possesses the attractor topologically equivalent to the attractor in the black box.
- All realizations possess the same dynamical and statistical properties as the signals at the output of the black box.

The problem of global reconstruction is ill-posed and nonunique. So, although it has no theoretically justified solution, there will be an infinite number of various approximate solutions. The most challenging problem is to select a proper form of evolution equations. However, if something is known about the contents of the black box, the task dramatically simplifies. Recently, methods were developed to reconstruct even extended systems in the form of partial differential equations and delay differential equations of specific forms [11, 12, 13].

In our case, by construction, we know *everything* about the generating system, *everything* about the receiving system, also the way the reflected signal enters the receiver and the reflected signal itself. The only unknown parameters are the time delays and attenuation factors. Thus, in this special case the problem of reconstruction reduces to the numerical estimation of several pairs of unknown parameters, namely, attenuation and delay for each reflected signal.

If there were no time delays, the task would be most trivial and reduced to the definition of $M$ parameters linearly entering the equations by means of a least squares technique [14]. However, we do not yet know time delays and cannot apply this method straightforwardly.

To find time delays $\tau_i$ one can use statistical approaches like the function of mutual correlation. *Disadvantages* of the use of statistics for this purpose are that a large amount of data is required and this method still is not very reliable for more than two delays.

We propose to exploit again the reconstruction approach to define any number of time delays in the received signal. Assume without loss of generality that the form of the system modeling our situation is as follows:

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2, \ldots, x_N; \vec{\mu}) \\
&\cdots \\
\dot{x}_N &= f_N(x_1, x_2, \ldots, x_N; \vec{\mu}) \\
\dot{y}_1 &= g_1(y_1, y_2, \ldots, y_L; \vec{\nu}) + a_0 x_1(t-\tau_0) + a_1 x_1(t-\tau_1) + \ldots + a_M x_1(t-\tau_M) \\
&\cdots \\
\dot{y}_L &= g_L(y_1, y_2, \ldots, y_L; \vec{\nu}).
\end{aligned}
\tag{14}
$$

Here, $\vec{\mu}$ and $\vec{\nu}$ are vectors of control parameters for each system which we assume to be fixed and known. Certainly, the way how the first system is coupled to the second one can be changed and depends only on the operator, so it is supposed to be known in any case. The values to be found are $a_i$ and $\tau_i$. We assume the attenuation factors $a_i$ to be explicitly related to the time delays $\tau_i$ through the lengths of the paths $r_i$. That is, each time delay corresponds to a unique attenuation.

Now suppose that we do not know all or some components of the parameter vectors $\vec{\mu}$ and $\vec{\nu}$. Let us apply global reconstruction to find them. It is clear that if time delays (and corresponding attenuations) are selected properly, the fitted parameters will be equal to the true values of them, and not equal otherwise. We propose to try systematically all possible combinations of distances $r_i$ in order to find a set of them which will result in correct fitting of the 'unknown' control parameters.

**Advantages:**

- Small data sets to process. If the number of 'unknown' parameters to fit is quite small (about 10 or less), the number of datapoints needed to estimate them by a least squares technique is quite small as well, being roughly 10 times larger than the number of unknown parameters [15].
- Straightforward estimation of delays and attenuations is sought avoiding the uncertainty of a bifurcation diagram approach.

In principle, in the ideal noiseless case this approach does not even need the plotting of bifurcation diagrams for the system.

**Disadvantages:**

- Possibly this could be a computing intensive task if all combinations of delays are to be considered.
- If the receiving system is near bifurcation, noise can lead to the effective shift of control parameter values and thus to their incorrect estimation.

In view of the second disadvantage, plotting of bifurcation diagrams of a noiseless system should still be helpful. In the case that parameters estimated by means of reconstruction take values close to bifurcation curves, one should be aware of the possible loss of precision. Thus, combination of bifurcation analysis and global reconstruction can provide a promising solution.

## 2.4 Other possible approaches

### 2.4.1 Controlling chaos

In general it is known that if a chaotic signal forces a chaotic system, so that coupling between the two chaotic systems is unidirectional, there is no possibility of obtaining a periodic attractor in a forced (receiving, in our terms) system. However, if some additional loop is applied to the receiver, it is possible to stabilize some selected unstable (saddle) periodic orbit embedded in a chaotic attractor. As a result, chaotic forcing applied to a chaotic system with such a feedback will produce a periodic attractor. This approach is in fact reduced to a recently introduced ideas of 'controlling chaos' [16, 17]. This idea was only mentioned during the Study Group and not developed to some explicit method to extract parameters of interest from the resulting periodic signal.

### 2.4.2 Neural networks

Alternatively, multi-layer neural networks can be used for pattern recognition of the received signals, *e.g.* [18, 19]. In this approach the device used to recognise the system parameters is a network of neurons or units modelling the relationships between the inputs and outputs. The network of neurons can be trained to associate the signals with the system parameters. The robustness of this approach largely depends on the quality of the network training.

# 3 Summary and conclusions

During the work of the group, problems of applications of chaotic signals for sonar techniques were discussed from the viewpoint of the use of bifurcation theory for signal recognition and processing. In this context, two suggested configurations of transmitter-receiver system were considered, namely, when transmitter and receiver are the same device (feedback scheme), and when they are separated (parallel scheme). It was revealed that the parallel scheme is preferable for the aims specified, since it is more secure than the feedback scheme. As a possible way to use bifurcation analysis, the use of 'anticipating synchronization' was suggested and its generalisation to accommodate multiple delays should open up a new avenue in tackling the problem. Some disadvantages of the bifurcation approach were revealed and discussed. The use of alternative complementing methods was proposed to improve the quality of the bifurcation approach. Another promising tool seems to be reconstruction techniques. Finally, two other approaches were proposed that might be suitable for sonar problems.

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