The vibrating tuning fork fluid density tool

Problem presented by

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Problem statement

Nan Gall are developing a densitometer for application down oil wells, based on a transducer supplied by Solartron. The transducer is not designed for such environments and the Study Group was asked to consider the following questions. What is the best mode of operation, in particular to ensure that the results are independent of viscosity? What are the effects of enclosing the device in the pressure rated housings that are required for deployment in oil wells? Finally, how does the device respond to an inhomogeneous fluid?

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1 Problem description

1.1 Circulated description

The following description was prepared by Alan Fleming of Nan Gall and circulated in advance of the Study Group.

> The problem involves a transducer supplied to Nan Gall by Solartron. It is used for measuring fluid density and viscosity. Nan Gall would like to apply this transducer for down oil-well applications. The transducer has not been used down-hole before, though it has been used in petroleum processing plants. Nan Gall buy only the transducer without Solartron's electronics and software. This is because Solartron's electronics will not fit in our pressure rated housing, are not rated to down-hole temperatures and draw too much current for battery operation. For further background, see Solartron's website: http://www.solartronmobrey.com/density/7828.html. We do not have a confidentiality agreement with Solartron. However, they are aware that we are doing our own electronics development and research into down-hole applications of the transducer. They have told me that they have performed some simulation of the system in the past but have not released the results to us. The transducer is based on the principle that the resonant frequency of an element is dependent on the density of the fluid in which it is immersed. This is presumably because some of the fluid is dragged with the vibrating element altering the effective mass. The viscosity of the fluid applies a damping force to the system. The Q of the resonance therefore decreases with increasing viscosity.[1]
>
> A tuning fork design is used because it is immune to external sources of vibration. The tuning fork is excited by a driver piezoelectric element. The resulting motion of the tuning fork is sensed by a pick-up piezoelectric element. The voltage applied to the driving piezo is proportional to the stress applied to the tuning fork. If the pick-up piezo is open circuit, the voltage obtained from the pick-up piezo is proportional to the strain. Alternatively, if the pick-up piezo is short circuit, then the current output is proportional to rate of strain. Nan Gall's electronic circuit applies a phase shift to the signal from the pick-up piezo and amplifies it to a constant peak-to-peak level. This voltage is then applied to the driver piezo. Depending on the phase shift applied it is possible to vibrate the tuning fork 'on-resonance' or to either side of the resonance. For example, if the pick-up is short circuit then:
>
> $0°$ phase shift causes vibration on resonance.
>
> $+45°$ phase shift causes vibration at the 3 dB down point with higher frequency (driver is 45° in front of pickup).

[1] Recall that if a resonator gives maximum response at frequency f_M, and responses of *half* that maximum power at frequencies f_A, f_B either side of f_M, then the 'quality' of the resonance is $Q = f_M/(f_B - f_A)$. So a *lightly* damped system has a *high* Q.

$-45°$ phase shift causes vibration at the 3 dB down point with lower frequency (driver is 45° behind pickup).

The transducer manufacturers recommended operation at the upper 3 dB point for measuring density. Our experiments have verified that this is the case at least for fluids up to about 100 cP viscosity, with a very good linear fit between frequency and density. Note that a simple model based on the equation of a simple damped harmonic oscillator:

$$m\ddot{x} = F - v\dot{x} - kx \tag{1}$$

(where m is the inertia (mass of tuning fork plus dragged fluid), v is the damping constant related to viscosity, k is the spring constant and F is the applied force) predicts that the resonant frequency is independent of viscosity. The frequency of the upper 3 dB point would appear to depend on viscosity. Maybe this model is invalid because the volume of fluid dragged by the tuning fork is dependent on viscosity.

The questions that we would like to be addressed by the study group are the following:

1. **What is the best way to operate the transducer to determine fluid density? Why and to what degree is the frequency of the upper 3 dB point independent of viscosity?**
2. **If the transducer is mounted in a cylindrical housing, how will this affect its operation?** The transducer will be mounted inside a housing smaller than that recommended by the transducer manufacturer, due to restrictions of running in an oil well. This affects the resonant frequency.

1.2 Notes of presentation to the Study Group

Paul Moseley (Smith Institute) presented the problem on behalf of Nan Gall Technology. The company would like to be able to measure the density and viscosity of mixtures of oil and water. Capacitive methods are not accurate when water fractions exceed 40%, and radioactivity-based methods are beset with regulatory problems: hence the interest in this vibration-based method.

The geometry of the 'tuning fork' is roughly as shown in Figure 1: the tines (prongs in common parlance) are prismatic, with segments of a circle as cross-section. The steel base in which they are mounted has diameter 17.5 mm, and the proposed housing would have an inner diameter of 26 mm.

A third question was added to 1. and 2. above:

3. **How is the device affected by inhomogeneous fluid ?**

Ellis Cumberbatch asked what accuracy is required, but this was not known. The paper [2] was referred to, in which a somewhat similar device was subject to mathematical analysis.

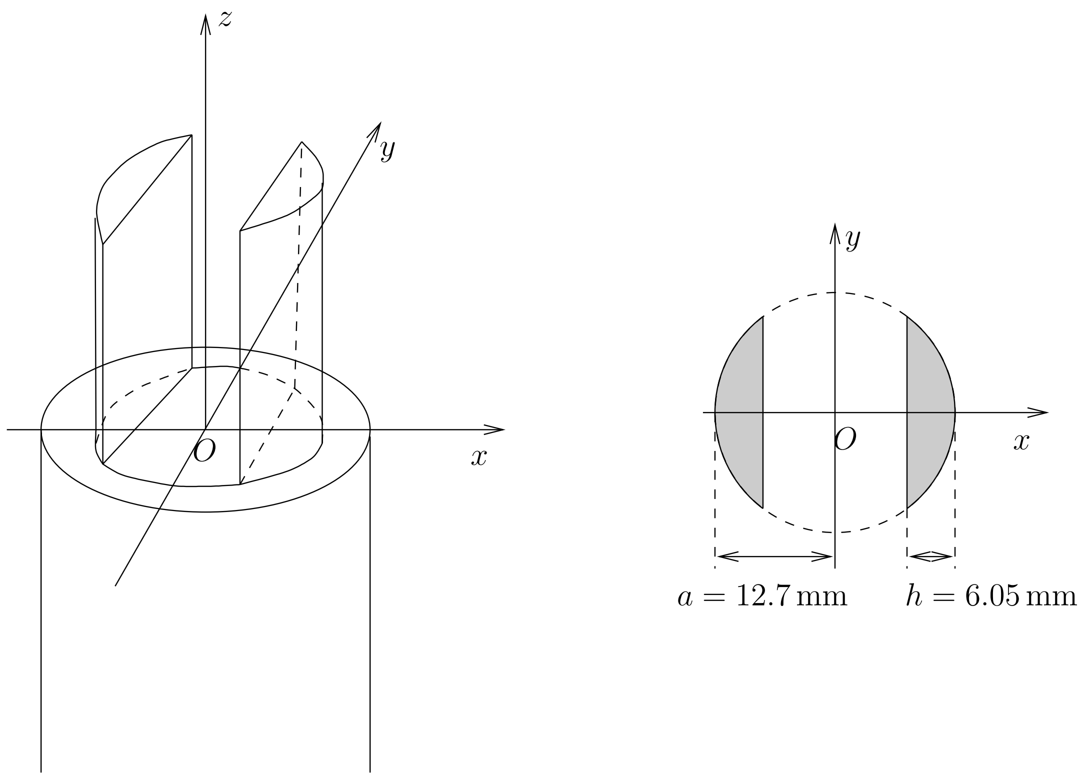

Figure 1: Diagram of tuning fork densitometer.

The fluid density ρ is calculated from the frequency f_B at the upper 3 dB point by the formula (given in [1])

$$\rho = K_0 + K_1\tau_B + K_2\tau_B^2, \tag{2}$$

where $\tau_B = 10^6/f_B$ is the period in microseconds, and K_0, K_1, K_2 are constants. Certain corrections are then applied to allow for temperature, fluids of higher viscosity *etc.*

2 Basics of the model

We shall use a coordinate system as in Figure 1, and we shall sometimes consider the full 3-dimensional motion, and sometimes motion in a 2-dimensional cross-section in the (x, y)-plane. The tines are treated as conventional bending beams, subject to fluid loading, so if $u(z, t)$ denotes the displacement of the centroid of the beam cross-section in the x-direction, then

$$m_s \frac{\partial^2 u}{\partial t^2} = -B\frac{\partial^4 u}{\partial z^4} + \int_D \sigma_{nx}\, ds, \tag{3}$$

where m_s denotes the mass per unit length of the steel, B the bending stiffness of the beam, D the circumference of the cross-section, and σ_{nx} the x-component of the normal stress on the beam surface.

We begin by comparing the orders of magnitude of the different physical effects that will be involved in σ. The typical order of magnitude of the resonant frequency involved is $f = 1500\,\mathrm{Hz}$, and as usual we work with $\omega = 2\pi f \approx 10^4\,\mathrm{rad/s}$.

Fluid inertia: For a segment of steel (density ρ_s) of width w and thickness h moving transversely in a fluid as illustrated in Figure 2, the ratio of the added mass of fluid per unit length to the mass per unit length of the segment itself is of the order of

$$\frac{\rho w^2}{\rho_s w h} \sim \frac{1}{2}. \tag{4}$$

This is therefore an important effect, as one would expect. The concept of added mass

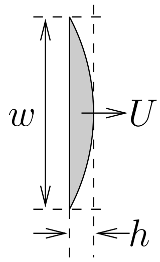

Figure 2: A segment moving in fluid.

is explained in the appendix, but briefly, if a rigid body of mass M is moved with acceleration $\dot{\mathbf{U}}$ through fluid of density ρ, then the force that has to be applied to it is $(M + M_a)\dot{\mathbf{U}} +$ (viscous drag *etc*): M_a is called the 'added mass', and is proportional to ρ and to the cube of the linear dimensions of the body (or the square for 2-dimensional flow).

Viscosity: We shall see later that the viscous boundary layer thickness is

$$\delta = \sqrt{\frac{\mu}{\rho\omega}}, \tag{5}$$

where μ is the dynamic viscosity and ω the frequency. Computing for figures given in [1] we find that this is about 0.3 mm, so $\delta/h \approx 0.05$. Hence viscous effects are important, but not as important as inertia.

Compressibility: The acoustic wavelength at the resonant frequency ω is about 1 m, which is much larger than any of the linear dimensions involved, so we shall neglect compressibility.

Nonlinearity: The amplitude of vibration involved is estimated by Solartron to be at most 10 μm, which is small compared to the boundary layer thickness δ, so we shall neglect nonlinear effects here. One nonlinear effect mentioned in discussions is shedding of vortices from the sharp edges of the vibrating tines. It is not clear how significant this will be — the edges are not razor sharp — but it is undoubtedly an effect that would have to be included in a more complete analysis, though we shall omit it here.

In conclusion, we shall use a linear, incompressible model, with fluid inertia and viscosity included. The general picture to have in mind is described by Batchelor [6, §5.13]: there is an oscillating viscous boundary layer on the tines, and outside that an oscillating inviscid flow in the rest of the fluid.

3 Vibration mode shape

In the vibration mode used in the device, the tines vibrate in antiphase in their lowest mode (as in a musical tuning fork), with the result that they do not exert any resultant force or moment on the base. We wish now to address briefly how the displacement $u(z,t)$ varies with z. If the tines were to vibrate as classical clamped-free bending beams (*i.e.* clamped at the end $z = 0$ and free at the end $z = l$) then (at least *in vacuo*) the classical theory [3, Ch VIII] would apply and the resonant frequencies ω_n would be given by $B(m_n/l)^4 = m_s\omega_n^2$ with dimensionless wavenumbers m_n that are the positive solutions to $\cos m_n \cosh m_n + 1 = 0$. The lowest mode corresponds to the solution $m_1 \approx 1.875$. For a parabolic segment[2] of width w and height h one finds that the cross-sectional area A, and the second moment of area I are given by

$$m_s/\rho_s = A = \tfrac{2}{3}wh, \quad B/E = I = \tfrac{8}{175}wh^3. \tag{6}$$

Hence the frequencies are given by

$$\omega_n^2 = (E/\rho_s)\tfrac{12}{175}h^2(m_n/l)^4. \tag{7}$$

Taking the figures in [1], we have $\rho_s = 7960\,\mathrm{kg/m^3}$, and Young's modulus E in the range 190–210 GPa, $h = 6.05\,\mathrm{mm}$, and $l = 45\,\mathrm{mm}$. This gives a lowest frequency f_1 in the range 2150–2260 Hz *in vacuo*, with $f_1 = 2206\,\mathrm{Hz}$ for $E = 200\,\mathrm{GPa}$. However, if $\rho = 0$ is used in (2) to estimate the natural frequency *in vacuo*, one finds it is 1908 Hz. This discrepancy, with the actual frequency *lower* than that of a clamped-free bending beam, is probably due to the lower end of the tine not being perfectly clamped. (This of course is necessary, otherwise it could not be driven.) In operation of course, E depends on temperature and this has to be allowed for in the coefficients.

We then have to address two further questions about the mode shape:

1. *How is the mode shape affected by the fluid?*
   If the fluid were constrained to move in independent 2-dimensional slices $z =$ constant, then the added mass per unit length would be constant along the tine, and the mode shape would not be affected: only the frequency would be reduced. However, this is clearly not the case for 2 reasons. First, the flow around the end of the tine violates the 2-dimensional assumption. Second, since the tine bends, the flow got by stacking up scaled copies of the 2-dimensional potential flow is not irrotational. We conclude that the added mass per unit length will certainly vary along the length of the tine, as it will depend on the whole 3-dimensional flow. Therefore, the fluid will to some extent affect the mode shape.

2. *How is the vibration mode shape affected by the method of forcing?*
   We do not have details of the method of forcing, but in outline, it is as illustrated in Figure 3. On the drive side, longitudinal strains are induced in the piezoelectric material, and since it is off-centre with respect to the tine axis, it exerts a torque

[2] The parabolic segment is an approximation to the circular, easier to calculate for. There are exact formulae for a circular segment, given for instance in [4]. But for our dimensions $a = 12.7\,\mathrm{mm}$, $h = 6.05\,\mathrm{mm}$ (and $w = 2\sqrt{h(2a-h)}$), the formulae (6) give the value of I/A accurate to about 1%.

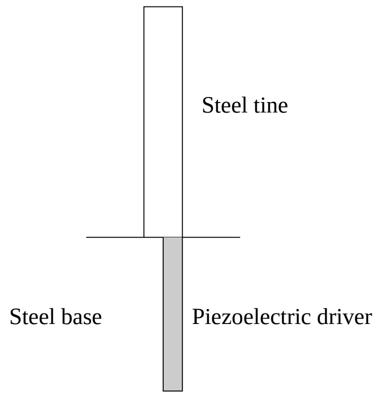

Figure 3: Diagram of forcing mechanism.

on the tine, thereby exciting it in bending. On the pick-up side, a physically identical arrangement results in longitudinal strain in the piezoelectric material, giving rise to the output signal. If the data in [1] are representative, then the facts that the system is forced within 45° of resonance, and has a high Q, imply that the forcing frequency is close to the resonant frequency. Hence the mode shape in the *forced* vibration is *close* to the resonant mode shape — though of course it cannot be identical to it. The details of the analysis of the drive mechanism and its effect on the vibration mode shape would require knowledge of the geometry of the mechanism, and also the stiffness of the piezoelectric element.

In spite of these questions being unanswered, we can still proceed to a lumped parameter model, but these considerations do indicate that certain 'constants' in the model will in fact have dependencies on frequency that we are neglecting.

4 Lumped parameter model

We now proceed to a model more on the lines of (1). We shall let F be the force applied by the driver piezo to the base of the driven tine, and x be the displacement measured by the pick-up piezo. Since we are considering a linear model, and are looking only at the resonant behaviour, we shall let $F = \mathrm{Re}(F_c e^{i\omega t})$ and $x = \mathrm{Re}(x_c e^{i\omega t})$, so F_c and x_c are the complex amplitudes of the force and displacement. Then we shall show that the basic model takes the form

$$F_c = -m_0\omega^2 x_c - \rho V_1 \omega^2 x_c + (i\omega)^{3/2}\sqrt{\mu\rho}A_1 x_c + v_0 i\omega x_c + k x_c. \qquad (8)$$

The first term here represents the tine inertia, m_0 being the effective mass of the tines, so it would be $m_0\ddot{x}$ in the time domain. In the second term V_1 is an effective volume, and ρV_1 the effective added mass of fluid, so this would be $(\rho V_1)\ddot{x}$ in the time domain. The third term we shall treat in a little more detail below in section 4.1. The fourth term $v_0 i\omega x_c$ is the intrinsic damping in the tuning fork, and would correspond to a damping term $v_0\dot{x}$ in the time-domain, and the last term represents the elastic stiffness of the tines and would be just kx in the time domain.

4.1 Oscillating boundary layer

Much the simplest example of an oscillating boundary layer is an infinite half space $z > 0$ of fluid, with the boundary plane $z = 0$ oscillated in the x-direction with displacement $x_c e^{i\omega t}$. The part of the actual tine that this is most closely analogous to is the flat end face. The fluid velocity then is $\mathbf{u} = (u_c(z)e^{i\omega t}, 0, 0)$ and the Navier-Stokes equations become simply $\rho i\omega u_c(z) = \mu d^2u_c/dz^2$. The velocity at $z = 0$ is $u_c(0) = x_c i\omega$, so the solution tending to 0 as $z \to +\infty$ is

$$u_c(z) = x_c i\omega \exp(-\sqrt{i\omega\rho/\mu}\, z) = x_c i\omega \exp(-\sqrt{i}\, z/\delta). \tag{9}$$

This analysis is the origin of the formula (5) for the boundary layer thickness δ. From this, the shear stress on the boundary surface is calculated as

$$(\sigma_{xz})_c = \mu \left.\frac{\partial u_c}{\partial z}\right|_{z=0} = -(i\omega)^{3/2}\sqrt{\mu\rho}\, x_c. \tag{10}$$

For a general body shape, a result with this same dependence on ω, ρ and μ will be obtained, provided that $|x_c| \ll \delta$. The reason for this is essentially that the flow outside the boundary layer is the inviscid flow that gives rise to the added mass term: this matches the normal velocity of the tines, but does not match the tangential velocity. Hence the flow relative to the tines inside the boundary layer just has to match to the tangential shear between the tines and the outer flow, and therefore has just the form analysed here. The general theory of oscillatory boundary layers is discussed by Batchelor in [6, §5.13], and other relevant references are [10], [11].

4.2 Phase angle control

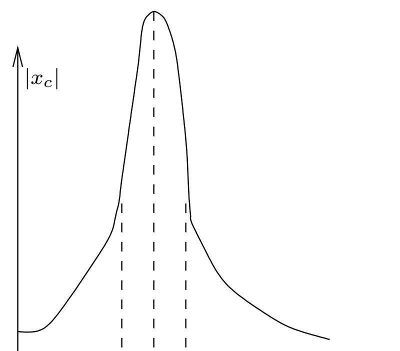

Figure 4: Response diagram (schematic).

In operation, the system electronics includes a frequency controller that adjusts the frequency until a certain (adjustable) phase relation obtains between drive and pickup. Taking the case where the pickup is short-circuit, a phase angle Δ corresponds to the force F_c being Δ ahead of the velocity $i\omega x_c$, so

$$\arg(F_c) = \arg(e^{i\Delta} i\omega x_c). \tag{11}$$

The points $\Delta = -\frac{\pi}{4}, 0, +\frac{\pi}{4}$ correspond to points A, M, B, on the response diagram in Figure 4.

5 Nan Gall's questions

With this model we can begin to answer Nan Gall Technology's questions.

1. **What is the best way to operate the transducer to determine fluid density ? Why, and to what extent, is the upper 3 dB point ($f_B = \omega_B/2\pi$) independent of viscosity ?**

From the model above, the equation determining the frequency ω for a given phase shift Δ is

$$\tfrac{1}{2}\pi + \Delta = \arg(e^{i\Delta} i\omega) = \arg(F_c/x_c) = \arg(-m_0\omega^2 - \rho V_1\omega^2 + (i\omega)^{3/2}\sqrt{\mu\rho}\, A_1 + v_0 i\omega + k). \tag{12}$$

So if we want the frequency ω to be approximately independent of fluid viscosity then we must choose Δ so that the third term has no influence. This will be when $\Delta = +\pi/4$, so that the third term already has the required phase angle $\frac{1}{2}\pi + \Delta = 3\pi/4$. The remaining four terms must then also have a phase angle of $3\pi/4$, which requires

$$-m_0\omega_B^2 - \rho V_1\omega_B^2 + k = -v_0\omega_B. \tag{13}$$

Writing this in the form

$$\rho = \frac{1}{V_1}\left(-m_0 + \frac{v_0}{\omega_B} + \frac{k}{\omega_B^2}\right), \tag{14}$$

we then have exact agreement with the form of the Solartron formula (2).

Moreover, at $\Delta = -\pi/4$ we shall have

$$-m_0\omega_A^2 - \rho V_1\omega_A^2 + k = v_0\omega_A + \omega_A^{3/2}\sqrt{2\mu\rho}A_1, \tag{15}$$

and this will be the basis of the method of estimating viscosity. Essentially the difference $\omega_B - \omega_A$ is approximately proportional to $\sqrt{\mu}$ as stated in [1].

2. **If the transducer is mounted in a cylindrical housing, how will this affect its operation ?**

The radius of the circumcircle of the tines is 12.7 mm, and the inner radius of the proposed housing is 26 mm, as illustrated in Figure 5. With a boundary layer thickness of 0.3 mm, the viscous term is unaffected by the housing — it is the added mass term ρV_1 that is going to be altered. It will certainly be increased by this constraint on the flow, and the question is: *How much ?* The increase could be calibrated, calculated or estimated. To *calibrate* it, one would simply measure the resonant frequency of the device in its housing, for a variety of fluids of known density, and fit to a formula of the type above. One would expect that the constants K_0, K_1, K_2 would all be decreased somewhat, and in the same proportion as each other. To *calculate* the change, one would

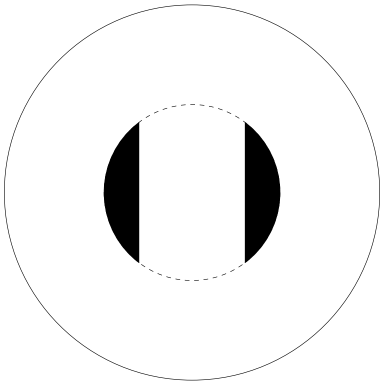

Figure 5: Tines in housing.

need to do a 3-dimensional added mass calculation, with some assumed bending shape of the tine. Since the added mass can be represented in a variational form, as mentioned in the appendix, it would not be too difficult to carry out this calculation accurately. Alternatively, if we just want an *estimate*, we can proceed more simply.

Consider the motion in a 2-dimensional cross-section. The motion of a single tine will generate a fluid motion that is dipole in the far field. The motion of both tines in antiphase will generate a motion that is quadrupole in the far field. Obviously the outer boundary is not really in the far field since $b/a \approx 2$, but still the simplest analogous question to ask is: *How is the added mass for a quadrupole of radius a affected by a concentric housing of radius b ?* For this, consider a circle of radius a with a radial velocity $U \cos 2\theta$ on its circumference as illustrated in Figure 6. (This is a suitably scaled version of the velocity field on the circle $r = a$ due to a point quadrupole at the origin: it is just intended to be generally similar to that due to the vibrating tines.) In an unbounded

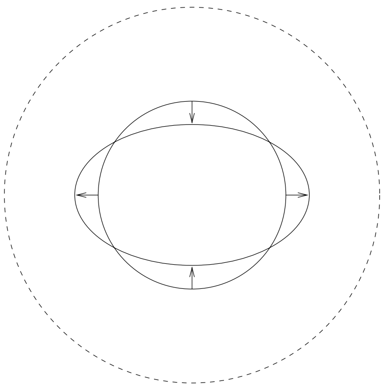

Figure 6: Quadrupole-style imposed motion on surface of cylinder.

fluid, the resulting flow is represented by the complex potential $w = -Ua^3/2z^2$, and the added mass corresponding to this can be calculated as $\frac{1}{2}\pi\rho a^2$. With a circular housing of radius b in place, the potential is modified to

$$w = -\frac{Ua^3}{2(b^4 - a^4)} \left(\frac{b^4}{z^2} + z^2 \right), \tag{16}$$

and the added mass is increased to

$$\tfrac{1}{2}\pi\rho a^2\left(\frac{b^4+a^4}{b^4-a^4}\right). \tag{17}$$

Hence with $b/a \approx 2$ this indicates that the added mass will increase by about 12%. In terms of the parameters in the basic equation (2), this would reduce the constants K_0, K_1, K_2 by 12%.

3. What happens in an oil-water mixture? Will the time-average of the estimated density give the true mean density?

The preceding analysis has assumed throughout that we have a homogeneous fluid around the tines. In a 'slugging' 2-phase flow, where slugs of oil (lighter) and water (denser) pass alternately, the device could give a successful indication of the duration of each phase. However, when an inhomogeneous fluid is present, in the form of oil and water, there are immediately various complications. Every oil-water interface will have oscillatory viscous boundary layers on each side, because of the different fluid densities, as shown in the appendix. If the fluid near the tines is reasonably well-mixed (*e.g.* with droplets up to perhaps a few mm in diameter) then it should be possible to estimate its mean density reasonably well by the device. However, even in the well-mixed case, viscosity measurement is likely to be a problem: the viscous effects depend on the boundary layer on the tines, and so we run into questions of whether that boundary layer is in the oil or in the water, which in turn will depend on whether oil or water preferentially sticks to the tines.

If the fluid is not well-mixed, then further potential problems may arise —

1. The frequency controller may fail to lock on to the required phase relationship, because the instantaneous value of the added mass is continually changing due to the flow, and the frequency controller is trying to lock on to this changing frequency. Unless the flow is particularly fast, it should be possible to cover this within the electronics, but it is a point to be borne in mind.

2. When a density reading is acquired, it will be representative of a spatially weighted average density near the tines. But the water fraction in the housing will not necessarily be representative of the net water fraction in the flow: firstly because the flow in the well will be in one of the various possible 2-phase flow regimes depending on the speed, inclination and relative proportions of the fluids; and secondly because the device itself will disturb that flow, and so there is no guarantee that the fluid in the housing is a representative sample of the total fluid mixture present. This problem is likely to be particularly acute if the well is not vertical, because of the tendency of the water and oil to stratify.

On a more positive note, we can be sure that any *increase* in fluid density above that of oil indicates that *some* water is present. (This is a consequence of point 7 in the discussion of added mass in the appendix.) One suggestion to try to ensure that the fluid in the housing is a representative sample is to have something that artificially mixes the flow before it enters the device. If there is gas present as well as water and oil, the situation will of course be further complicated.

A Appendix: Added mass

The concept of added mass, also called virtual mass, can be approached either in terms of accelerated motion or impulsive motion, and some standard descriptions are given by Lamb [7, Chapter VI], Milne-Thomson [8, 9.22], Batchelor [6, §6.4], and Lighthill [9, Section 8.3]. Briefly, if a body of mass M is at rest in an unbounded region of incompressible fluid of density $\rho(\mathbf{x})$ which is also at rest, then in order to suddenly impart a velocity $\mathbf{U}$ to the body, the impulse that must be applied is $(M + M_a)\mathbf{U}$: $M\mathbf{U}$ is the impulse that would have to be applied to get the mass moving *in vacuo*, and $\mathbf{I} = M_a\mathbf{U}$ is the *additional* impulse that has to be applied to get the fluid moving compatibly with the motion of the body — this M_a is the added mass. Equivalently, if (again starting from rest) the body is given an acceleration $\dot{\mathbf{U}}$, the force that must be applied to it is $(M + M_a)\dot{\mathbf{U}}$, just as if the actual mass M had been increased by M_a. The references mentioned apply to the case where ρ is uniform, but we shall extend the description here to the case of an *inhomogeneous* fluid where each fluid element retains its initial density, so $D\rho/Dt = 0$, and consequently the velocity field $\mathbf{u}$ obeys $\nabla.\mathbf{u} = 0$ as usual. This is what we need for the case where both oil and water are present, as in Nan Gall Technology's third question. Since the fluid is treated as incompressible, the motion of the body is transmitted to all parts of the fluid instantaneously. However, since we are only looking at the *initial* motion, before any viscous boundary layers are established or any separation takes place, the effects of viscosity are confined to a vanishingly small layer on the surface of the body. Added mass calculations are therefore carried out on the basis of *inviscid* flow.

The initial motion of the fluid when the body begins moving impulsively with velocity $\mathbf{U}$ can be characterized by Kelvin's minimum energy theorem: *If any number of points of a dynamical system are suddenly set in motion with prescribed velocities, the kinetic energy of the resulting motion is less than that of any other kinematically possible motion which the system can take with the prescribed velocities.* (See for instance [5, §108].) So the initial motion of the fluid ('dynamical system') is such as to minimize its kinetic energy subject to the incompressibility condition ('kinematically possible motions') and the constraint on the normal velocity ('prescribed velocities') at the surface of the body ('points of the dynamical system suddenly set in motion').

If the fluid is of uniform density, then this minimum occurs when $\nabla\wedge\mathbf{u} = 0$, and (for a multiply-connected fluid region) when there is no circulation.[3] So we have potential flow, and this is the usual form in which one meets Kelvin's minimum energy theorem. In the *inhomogeneous* case, the minimum occurs when $\nabla\wedge(\rho\mathbf{u}) = 0$. In fact $\rho\mathbf{u} = -\nabla P$ where P is the pressure impulse that occurs when the motion starts, so $\int_C \rho\mathbf{u}.\mathbf{dx} = 0$ for any closed curve C in the fluid, and this is the analogue for inhomogeneous flow of the usual circulation condition. A consequence of this should be noted: the condition $\nabla\wedge(\rho\mathbf{u}) = 0$ implies that on an interface between fluids of different densities ρ_1, ρ_2, the *tangential* velocities $\mathbf{u}_{t1}$ and $\mathbf{u}_{t2}$ are related by $\rho_1\mathbf{u}_{t1} = \rho_2\mathbf{u}_{t2}$. Hence there is in general a *discontinuity* in tangential velocity across the interface, and so a boundary layer will develop there in viscous flow. This is an example of the creation of vorticity

[3] We use $\nabla\wedge\mathbf{u}$ to denote the 'curl' or 'rotation' of the vector field $\mathbf{u}$: it is sometimes also written as $\nabla \times \mathbf{u}$.

by inhomogeneity, which is discussed in [12, Ch.1 Sect.4].

There are various points to note about added mass:

1. For an *asymmetric* body, or an asymmetric density distribution $\rho(\mathbf{x})$ in the fluid, the impulse $\mathbf{I} = M_a\mathbf{U}$ need not be parallel to $\mathbf{U}$, so M_a is not a scalar, but a tensor — which we shall think of as a 3×3 matrix. It is in fact symmetric, with positive eigenvalues, like an inertia tensor.

2. The kinetic energy imparted to the fluid when the body is started into motion impulsively with velocity $\mathbf{U}$ is $T = \frac{1}{2}\mathbf{U}.\mathbf{I} = \frac{1}{2}\mathbf{U}.M_a\mathbf{U}$.

3. If the body is axisymmetric about a line L, and $\mathbf{U}$ lies along L, then $M_a\mathbf{U}$ will also lie along L, so L is a principal axis of M_a.

4. If $\mathbf{U}$ lies in a plane of symmetry of the body, then $M_a\mathbf{U}$ also lies in that plane.

5. M_a is proportional to the cube of the linear dimensions of the body in 3-dimensional flow, or the square in 2-dimensional.

6. In the homogeneous case, M_a is directly proportional to the density ρ of the fluid.

7. In the inhomogeneous case, if $\rho_1(\mathbf{x})$ and $\rho_2(\mathbf{x})$ are two density distributions with $\rho_1(\mathbf{x}) \geq \rho_2(\mathbf{x})$ everywhere, then the added mass for ρ_1 is at least that for ρ_2, $M_{a1} \geq M_{a2}$. (By this inequality of symmetric matrices, we mean as usual that $\mathbf{U}.M_{a1}\mathbf{U} \geq \mathbf{U}.M_{a2}\mathbf{U}$ for *all* vectors $\mathbf{U}$.)

It is perhaps worth demonstrating points 1, 2 and 7 here, since the proofs given in the cited literature depend on assuming homogeneous fluid. If the body is impulsively set in motion with velocity $\mathbf{U}$ and then is immediately given an acceleration $\dot{\mathbf{U}}$, the force required for this is $M_a\dot{\mathbf{U}}$, and so the rate at which that force is doing work is $\mathbf{U}.M_a\dot{\mathbf{U}}$. This must be equal to the rate of change of kinetic energy of the fluid. However, the kinetic energy of the fluid is certainly *some* positive quadratic form in $\mathbf{U}$, say $\frac{1}{2}\mathbf{U}.M_1\mathbf{U}$ where M_1 is symmetric and positive. Hence

$$\mathbf{U}.M_a\dot{\mathbf{U}} = \frac{d}{dt}(\tfrac{1}{2}\mathbf{U}.M_1\mathbf{U}) = \mathbf{U}.M_1\dot{\mathbf{U}}. \tag{18}$$

This holds for any $\mathbf{U}$ and $\dot{\mathbf{U}}$, so $M_a = M_1$ is indeed symmetric and positive, which proves points 1 and 2. Point 7 follows from Kelvin's minimum energy theorem, since for *any* velocity distribution $\mathbf{u}$ in the fluid region F,

$$T_1 = \int_F \tfrac{1}{2}\rho_1|\mathbf{u}|^2 \geq \int_F \tfrac{1}{2}\rho_2|\mathbf{u}|^2 = T_2. \tag{19}$$

The constraints on $\mathbf{u}$ (incompressibility and compatibility with the motion of the body) do not depend on ρ, and hence $\frac{1}{2}\mathbf{U}.M_{a1}\mathbf{U} = \min_{\mathbf{u}}(T_1) \geq \min_{\mathbf{u}}(T_2) = \frac{1}{2}\mathbf{U}.M_{a2}\mathbf{U}$. (This argument, in a more general context, is due to Rayleigh [3, §79].)

Generalizations of the concept of added mass can be made to cases where there are other bodies present, including rigid boundaries, and to cases where the moving body is not rigid. To describe this latter generalization briefly, suppose the body moves with local normal velocity $Uf(\mathbf{p})\mathbf{n}$ at point $\mathbf{p}$ on the surface. In fact, we could imagine the interior of the body to be constrained by a light frictionless ingenious device operated by a plunger such that when the plunger displacement is ϵ, the normal displacement of point $\mathbf{p}$ on the body surface is $\epsilon f(\mathbf{p})\mathbf{n}$. Then immerse the body in fluid of density ρ and ask what impulse I has to be applied to the plunger to set the surface in motion with velocity $Uf(\mathbf{p})\mathbf{n}$? This impulse I will be proportional to U, and so if we write $I = M_a U$ then M_a has the dimensions of mass and is the added mass associated with this velocity distribution. It is this concept that we are using when discussing the case of the quadrupole motion, where we take $f(\mathbf{p}) = \cos 2\theta$ at the point $\mathbf{p} = (a\cos\theta, a\sin\theta)$.

A.1 Example of an added mass calculation in inhomogeneous fluid

Suppose we have 2-dimensional flow, with a circular body of radius a immersed in fluid of density ρ_0 extending from $r = a$ to $r = b$, which is in turn surrounded by fluid of density ρ_1 extending from $r = b$ to infinity, as in Figure 7. To calculate the added mass

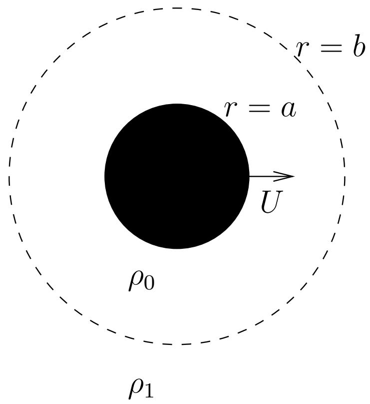

Figure 7: Circular body in inhomogeneous fluid.

in this situation, we must calculate the inviscid flow that is set up when the circular body of radius a suddenly starts moving with velocity U, which we may take along the x-axis. The pressure impulse P will clearly take the form $P = P(r)\cos\theta$ in polar coordinates, and the fluid velocity is given by $\rho\mathbf{u} = -\nabla P$, and must obey $\nabla.\mathbf{u} = 0$. This means that we have potential flow in both $a < r < b$, and $r > b$, so $P(r) = Ar + B/r$ in $a < r < b$, and $P(r) = C/r$ in $r > b$. To match at $r = b$ we need the pressure impulse continuous, $P(b-) = P(b+)$, and the *radial* component of velocity continuous, $P'(b-)/\rho_0 = P'(b+)/\rho_1$. And for compatibility with the motion of the circular body we need $P'(a) = -\rho_0 U$. This gives 3 linear equations to solve for A, B, C, from which the added mass M_a can be found. It is a scalar in this case obviously, and is given by

$$\frac{M_a - \rho_0\pi a^2}{M_a + \rho_0\pi a^2} = \frac{a^2}{b^2}\left(\frac{\rho_1 - \rho_0}{\rho_1 + \rho_0}\right). \tag{20}$$

This shows the expected features:

1. If $\rho_1 > \rho_0$ then $\rho_0\pi a^2 < M_a < \rho_1\pi a^2$, and M_a varies monotonically with b, tending to $\rho_1\pi a^2$ as $b \to a$, and $\rho_0\pi a^2$ as $b \to \infty$.

2. If $\rho_1 \to \infty$ then $M_a \to \rho_0 \pi a^2 (b^2 + a^2)/(b^2 - a^2)$, which is the value of the added mass for a circle of radius a surrounded by an annulus $a < r < b$ of fluid of density ρ_0 with a *fixed* outer boundary of radius b: the fluid of infinite density effectively forms a rigid boundary.

3. If $\rho_1 \to 0$ then $M_a \to \rho_0 \pi a^2 (b^2 - a^2)/(b^2 + a^2)$, which is the value of the added mass for a circle of radius a surrounded by an annulus $a < r < b$ of fluid of density ρ_0 with a *free* outer boundary of radius b.

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